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COGROWTH OF REGULAR GRAPHS

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ABSTRACT. Let $\mathcal{G}$ be a d-regular graph and T the covering tree of $\mathcal{G}$. We define a cogrowth constant of $\mathcal{G}$ in T and express it in terms of the first eigenvalue of the Laplacian on $\mathcal{G}$. As a corollary, we show that the cogrowth constant is as large as possible if and only if the first eigenvalue of the Laplacian on $\mathcal{G}$ is zero. Grigorchuk's criterion for amenability of finitely generated groups follows.

In this note, we shall relate the first eigenvalue of the Laplacian on a connected regular graph to the size of the kernel of the universal covering map. The main results have been proven in [C, G, P]. The proof presented here appears simpler; it depends on the explicit formula for minimal positive solutions of $\Delta F + \varepsilon F = -I$.

Let $\mathcal{G}$ be a connected simple graph with constant vertex degree $d \geq 3$, T be the universal covering tree of $\mathcal{G}$, and θ the covering map (i.e., θ is a vertex surjection of T on $\mathcal{G}$ that preserves adjacency and vertex degree). We let T and $\mathcal{G}$ denote the vertex sets of the corresponding graphs. Note that T has constant vertex degree d. Since T is connected, T may be considered a metric space with the usual graph metric δ ($\delta(x, y)$ is the length of the shortest path connecting x and y). For $x \in T$ and $n \geq 0$, let $[x] = \theta^{-1}(\theta(x))$ and $S_n(x) = \{y : \delta(x, y) = n\}$. For $x, y \in T$, note that

$$\limsup_{n\to\infty} |[y] \cap S_n(x)|^{1/n} = \inf\left\{\lambda > 0 : \sum_{z\in[y]} \lambda^{-\delta(x,z)} < \infty\right\}$$

and is thus independent of x and y. We call this number, $\operatorname{cogr}(T, \mathcal{G})$, the cogrowth constant of $\mathcal{G}$ in T.

For x, y vertices of a graph, we write xEy if x and y are connected by an edge. For $x, y \in T$, let

$$q(x, y) = \begin{cases} \frac{1}{d} & \text{if } xEy, \\ 0 & \text{otherwise.} \end{cases}$$

Note that q is the transition matrix of the simple random walk on T. Let $q^{(n)}$ denote the nth power of q. For $a, b \in \mathcal{G}$ and $x \in \theta^{-1}(a)$, since θ takes the

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simple random walk on T to the simple random walk on $\mathscr{G}$,

$$(1) \qquad p^{(n)}(a, b) = \sum_{y \in \theta^{-1}(b)} q^{(n)}(x, y)$$

where p is the transition matrix of the simple random walk on $\mathscr{G}$. We define $\Delta \equiv p - 1$, $G \equiv \sum_{n\geq 0} p^{(n)}$, and for $\varepsilon \geq 0$, $G^\varepsilon \equiv \sum_{n\geq 0} p^{(n)}/(1-\varepsilon)^{n+1}$. Similarly, we define Δ_T, F, and F^ε as above with p replaced by q.

Δ and Δ_T are the Laplacians on $\mathscr{G}$ and T, respectively. We call $\lambda_\mathscr{G}$ and λ_T the first eigenvalues of Δ and Δ_T, respectively. Since T is d-regular,

$$(2) \qquad \lambda_T = 1 - 2(d-1)^{1/2}/d$$

(see [DK]). Also,

$$(3) \qquad \lambda_\mathscr{G} = \sup\{\lambda : \exists f > 0 : \Delta f + \lambda f \leq 0\}$$

(see [DK] or [N]). It is true that $\lambda_\mathscr{G} \leq \lambda_T$ (see [N]).

For $\varepsilon \leq \lambda_T$, let $a(\varepsilon) = d(1-\varepsilon)/(d-1)$, $b = 1/(d-1)$, and $\sigma(\varepsilon) \leq \tau(\varepsilon)$ be the (real) roots of $t = a(\varepsilon) - b/t$. Note that

$$\sigma(\varepsilon) = \{d(1-\varepsilon) - [d^2(1-\varepsilon)^2 - 4d + 4]^{1/2}\}/2(d-1),$$
$$\tau(\varepsilon) = \{d(1-\varepsilon) + [d^2(1-\varepsilon)^2 - 4d + 4]^{1/2}\}/2(d-1).$$

In particular, $\sigma(\varepsilon)$ is increasing and $\tau(\varepsilon)$ is decreasing on $[0, \lambda_T)$.

Lemma. *For* $\varepsilon \in [0, \lambda_T)$, $F^\varepsilon(x, y) = \sigma(\varepsilon)^{\delta(x,y)}/(1 - \varepsilon - \sigma(\varepsilon))$.

Proof. Let $\varepsilon \in [0, \lambda_T)$. For $\lambda \in (\varepsilon, \lambda_T)$, there exists a function $f > 0$ such that $\Delta_T f + \lambda f \leq 0$. Let $v = -(\Delta_T f + \varepsilon f)/(1-\varepsilon)$ and $r = q/(1-\varepsilon)$. Note that $v > 0$ and $f = v + qf$. By induction, $f = \sum_{0\leq k\leq n} r^{(k)} v + r^{(n+1)} f \geq \sum_{0\leq k\leq n} r^{(k)} v$ since $f > 0$. Letting $n \to \infty$, $f \geq \sum_{k\geq 0} r^{(k)} v = (1-\varepsilon) F^\varepsilon v$. Since $v > 0$, F^ε exists.

By the symmetry of T, there exists a sequence $\gamma_0, \gamma_1, \ldots$ such that for any x, if $\delta(x, y) = n$ then $F^\varepsilon(x, y) = \gamma_n$. Since $(\Delta_T + \varepsilon)F^\varepsilon = -I$, it follows that $\gamma_{k+2} = a\gamma_{k+1} - b\gamma_k$ and $\gamma_1 - (1-\varepsilon)\gamma_0 = -1$. Let $r_k = \gamma_{k+1}/\gamma_k$, $\mu = a/2$, and $\nu = [a^2/4 - b]^{1/2}$. By the addition of angle formulae for hyperbolic functions, it is easy to verify that for all r_0 there exists θ, so that

$$r_n = \begin{cases} \mu + \nu\tanh(\theta + \rho n) & \text{if } r_0 \in (\sigma, \tau), \\ \mu + \nu\coth(\theta + \rho n) & \text{if } r_0 \notin [0, \tau], \\ r_0 & \text{if } r_0 \in \{\sigma, \tau\}, \end{cases}$$

where $\rho = \tanh(\nu/\mu)$.

Clearly, if $r_0 \neq \sigma$ then $r_n \to \tau$, and thus $\lim_{n\to\infty} \gamma_n^{1/n} = \tau$. It is easy to verify that $\lim_{n\to\infty} \gamma_n^{1/n}$ is increasing as a function of ε (since $p^{(n)} \geq 0$). Therefore $r_n \equiv \sigma$ since τ is decreasing. It follows that $F^\varepsilon(x, y) = c\sigma^{\delta(x,y)}$ and, since $\tau_1 - (1-\varepsilon)\gamma_0 = -1$, $c = 1/[1 - \varepsilon - \sigma]$. □

Theorem. (a) $\operatorname{cogr}(T, \mathscr{G}) \leq \{d(1-\lambda_\mathscr{G}) + [d^2(1-\lambda_\mathscr{G})^2 - 4d + 4]^{1/2}\}/2$,

(b) *If* $\lambda_\mathscr{G} \neq \lambda_T$ *then* $\operatorname{cogr}(T, \mathscr{G}) = \{d(1-\lambda_\mathscr{G}) + [d^2(1-\lambda_\mathscr{G})^2 - 4d + 4]^{1/2}\}/2$.

Proof. (a) If $\lambda_\mathscr{G} = 0$ then $\operatorname{cogr}(T, \mathscr{G}) \leq \limsup_{n\to\infty} |S_n(x)|^{1/n} = d - 1 = 1/\sigma(\lambda_\mathscr{G})$.

Let $\lambda_{\mathscr{G}} > 0$ and $\varepsilon \in (0, \lambda_{\mathscr{G}})$. As in the proof of the lemma, G^{ε} exists. Since

$$\sum_{z \in [y]} \sigma^{\delta(x, z)} = c \sum_{z \in [y]} F^{\varepsilon}(x, z) = cG^{\varepsilon}(\theta(x), \theta(y)) < \infty,$$

$\operatorname{cogr}(T, \mathscr{G}) \leq 1/\sigma(\varepsilon)$. Since $\sigma(\varepsilon)$ is increasing, the result follows by letting ε approach $\lambda_{\mathscr{G}}$.

(b) Let $\varepsilon \in [0, \lambda_T)$. If $\operatorname{cogr}(T, \mathscr{G}) < 1/\sigma(\varepsilon)$, then

$$G^{\varepsilon}(\theta(x), \theta(y)) = \sum_{z \in [y]} F^{\varepsilon}(x, z) = \sum_{z \in [y]} \sigma^{\delta(x, z)} < \infty$$

and thus G^{ε} exists. Fix $g \in \mathscr{G}$ and let $f(x) = G^{\varepsilon}(g, x)$. Clearly $\Delta_T f + \varepsilon f \leq 0$ and $f > 0$ and, therefore, $\varepsilon \leq \lambda_{\mathscr{G}}$. Assume $\lambda_{\mathscr{G}} \neq \lambda_T$ (and thus $\lambda_{\mathscr{G}} < \lambda_T$). If $\lambda_{\mathscr{G}} < \lambda_{\mathscr{G}} + \kappa \leq \lambda_T$, then $\operatorname{cogr}(T, \mathscr{G}) \geq 1/\sigma(\lambda_{\mathscr{G}} + \kappa)$. Since $1/\sigma$ is decreasing on $[0, \lambda_{\mathscr{G}}]$, $\operatorname{cogr}(T, \mathscr{G}) \geq 1/\sigma(\lambda_{\mathscr{G}})$. □

Corollary 1. *Let $\mathscr{G}$ be connected and d-regular. Then* $\operatorname{cogr}(T, \mathscr{G}) = d - 1$ *iff* $\lambda_{\mathscr{G}} = 0$.

Let A be a finitely generated discrete group with k generators, F the free group with k generators, ϕ the canonical mapping of F onto A, and $K = \ker \theta$.

The map ϕ induces a covering map θ from T onto $\mathscr{G}$ where T and $\mathscr{G}$ are the Cayley graphs of F and A respectively. As is well known, A is amenable iff $\lambda_{\mathscr{G}} = 0$ (see [K, DK, DG]).

By [P], $\lim_{n\to\infty} |K \cap S_{2n}|^{1/2n}$ exists.

Corollary 2. *A is amenable iff* $\lim_{n\to\infty} |K \cap S_{2n}|^{1/2n} = 2k - 1$.

References

[C] J. L. Cohen, *Cogrowth and amenability of discrete groups*, J. Funct. Anal. **48** (1982), 301–309.

[DG] Y. Derriennic and Y. Guivarc'h, *Theoreme de renouvellement pour les groupes moyennables*, C. R. Acad. Sci. Paris Ser. A **277** (1973), 613–615.

[DK] J. Dodziuk and L. Karp, *Spectral and function theory of combinatorial Laplacians*, Contemp. Math., vol. 73, Amer. Math. Soc., Providence, RI, 1988, pp. 25–40.

[G] R. I. Grigorchuk, *Symmetric random walks on discrete groups*, Multi-Component Random Systems (Dobrushin and Sinai, eds.), Dekker, New York and Basel, 1980, pp. 285–325.

[K] H. Kesten, *Full Banach mean values on countble groups*, Math. Scand. **7** (1959), 146–156.

[N] S. Northshield, *Gauge and conditional gauge on negatively curved graphs*, Stochastic Anal. Appl. **9** (1991), 461–482.

[P] A. T. Paterson, *Amenability*, Math. Surveys Monographs, vol. 29, Amer. Math. Soc., Providence, RI, 1988.

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