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INTEGRATING ACROSS PASCAL'S TRIANGLE

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Abstract. Sums across the rows of Pascal's triangle yield 2^n while certain diagonal sums yield the Fibonacci numbers which are asymptotic to ϕ^n where ϕ is the golden ratio. Sums across other diagonals yields quantities asymptotic to c^n where c depends on the directions of the diagonals. We generalize this to the continuous case. Using the gamma function, we generalize the binomial coefficients to real variables and thus form a generalization of Pascal's triangle. Integration over various families of lines and curves yields quantities asymptotic to c^x where c is determined by the family and x is a parameter. Finally, we revisit the discrete case to get results on sums along curves.

1. INTRODUCTION.

Sums of rows of Pascal's triangle are of course powers of 2. Of similar flavor, certain diagonal sums across Pascal's triangle yield the Fibonacci numbers which are, asymptotically, powers of the Golden Ration ϕ. It was noticed some time ago [3,5] that other diagonal sums yield things asymptotic to powers of various constants. We review this case in section 2 below.

A natural generalization is to use the gamma function to generalize the binomial coefficients to have two *real* variables and then to integrate over a family of lines. Specifically, given a function $F(x,y)$ homogeneous of degree 1 (F is C^2 and $F(kx,ky) = kF(x,y)$), we integrate across the curve $F(x,y) = r$ and find the result is asymptotic to $C\gamma^r$ (as r goes to infinity) for explicitly determined C and γ.

A consequence is that

$$\limsup_{F(i,j)\to\infty} \ln \binom{i+j}{i} / F(i,j) := \limsup_{r\to\infty} \sup\{\ln \binom{i+j}{i} / r : F(i,j) = r\} = \ln \gamma.$$

2. DISCRETE CASE.

Two well known facts about summing along rows and other lines across Pascal's triangle are

$$\sum_{i+j=n} \binom{i+j}{i} = 2^n,$$

and

$$\sum_{2i+j=n} \binom{i+j}{i} = F_{n+1} \sim C\phi^n$$

where F_n is the n-th Fibonacci number and ϕ is the "Golden ratio" 1.618... .

In a number of papers, generalizations have been considered. For example, Raab [5] was apparently the first to investigate sums

$$c_n := \sum_{ai+bj=n} \binom{i+j}{i}$$

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for integral and relatively prime a and b; see also the paper by Green [3]. For a generalization to sums over Pascal's triangle *mod 2*, see [4]. To understand the asymptotic behavior of the sequence (c_n), note that it satisfies the recurrence

$$c_{n+a+b} = c_{n+a} + c_{n+b}.$$

Thus (c_n) has a Binet type formula

$$c_n = \sum_i A_i \gamma_i^n \tag{1}$$

where $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ are solutions of the equation $x^{a+b} = x^a + x^b$. This equation has a unique solution γ of largest absolute value and, furthermore, that solution is positive and real (since it is possible to construct a directed graph whose (positive and 'aperiodic') adjacency matrix $M_{a,b}$ satisfies the Perron-Frobenius theorem and has characteristic equation $x^{a+b} = x^a + x^b$).

By the Binet type formula (1),

$$c_n \sim A\gamma^n$$

for some constant A.

An alternative way of looking at this is via generating functions. With the sequence c_n defined above, its generating function is then

$$\sum_n c_n x^n = \sum_n \sum_{ai+bj=n} \binom{i+j}{i} x^{ai+bj}$$

$$= \sum_{i,j} \binom{i+j}{i} (x^a)^i (x^b)^j = \frac{1}{1-x^a-x^b}$$

$$= \frac{1}{Q(x)(x-\sigma)} = \frac{P(x)}{Q(x)} + \frac{A}{\sigma - x}$$

where σ is the unique positive root of $1-x^a-x^b = 0$. Replacing x by σx and subtracting geometric series on right,

$$\sum_n (c_n\sigma^n - \frac{A}{\sigma})x^n = \frac{P(\sigma x)}{Q(\sigma x)}.$$

Letting $x \to 1$, the right side is finite and so
$\sum_n (c_n\sigma^n - \frac{A}{\sigma}) < \infty$ and thus $c_n\sigma^n \to A/\sigma$. Equivalently,

$$c_n \sim \frac{A}{\sigma}\sigma^{-n}.$$

To identify A, recall

$$\frac{1}{1-x^a-x^b} = \frac{P(x)}{Q(x)} + \frac{A}{\sigma - x}$$

and thus

$$1 = (1-x^a-x^b)\frac{P(x)}{Q(x)} - A\frac{1-x^a-x^b}{x-\sigma}.$$

Letting $x \to \sigma$, since $\sigma^a + \sigma^b = 1$,

$$1 = A \lim_{x\to\sigma} \frac{x^a - \sigma^a + x^b - \sigma^b}{x-\sigma} = A(a\sigma^{a-1} + b\sigma^{b-1}).$$

Then

$$c_n \sim \frac{A}{\sigma}\sigma^{-n} = \frac{\sigma^{-n}}{a\sigma^a + b\sigma^b}.$$

Let $\gamma = 1/\sigma$. Then γ is the unique positive root of $x^a + x^b = x^{a+b}$ and, finally,

$$\sum_{ai+bj=n} \binom{i+j}{i} \sim \frac{\gamma^{a+b}}{a\gamma^b + b\gamma^a}\gamma^n. \tag{2}$$

This equation extends for a, b positive rational.

Theorem 1. *For $r, s \in \mathbb{Q}^+$,*

$$\limsup_{n\to\infty} \gamma^{-n} \sum_{ri+sj=n} \binom{i+j}{i} = \frac{1}{c} \frac{\gamma^{r+s}}{r\gamma^s + s\gamma^r} \tag{3}$$

where γ is the unique positive root of $x^r + x^s = x^{r+s}$ and c is the smallest integer such that $cr, cs \in \mathbb{Z}$.

Proof. Let r, s be positive rational numbers. There are then three positive integers a, b, c such that $r = a/c$ and $s = b/c$. Let δ be the unique positive root of $x^a + x^b = x^{a+b}$. Then equation (1) implies

$$\limsup_{n\to\infty} \delta^{-cn} \sum_{ai+bj=cn} \binom{i+j}{i} = \limsup_{n\to\infty} \delta^{-n} \sum_{ai+bj=n} \binom{i+j}{i} = \frac{\delta^{a+b}}{a\delta^b + b\delta^a}.$$

Let $\gamma = \delta^c$. Then γ is the unique positive solution to $\gamma^{r+s} = \gamma^r + \gamma^s$ and, rewriting (2) with γ gives the result $\square$.

3. CONTINUOUS CASE.

The gamma function interpolates the factorial function (see [1] or [6]) and, as is customary, we may extend the binomial coefficients to real variables by the formula: for $x, y > 0$,

$$\binom{x+y}{x} := \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} = \prod_i \frac{(x+i)(y+i)}{(x+y+i)i}.$$

We shall integrate these generalized binomial coefficients over the level curves $F(x, y) = r$, where $F : \mathbb{R}^+ \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ ($\mathbb{R}^+ := [0, \infty)$). To have a result like (2), we assume that F is *homogeneous of degree 1*: for $k \geq 0$,

$$kF(x, y) = F(kx, ky). \tag{4}$$

Some examples are $ax + by$ ($a, b > 0$), $\sqrt{x^2 + y^2}$ and, more generally, $(ax^p + by^p)^{1/p}$ ($a, b > 0, p \geq 1$).

We must also make some further assumptions. Define an auxillary function $\psi(z) := F(z, 1)$ so that

$$F(x, y) = y\psi(x/y). \tag{5}$$

We shall assume that ψ is twice differentiable on $[0, \infty)$ and, for $z > 0$,

$$\psi(z) - z\psi'(z) \geq 0 \text{ , } \psi''(z) \geq 0 \text{ , } \psi(0) > 0 \text{ , and } \psi'(0) > 0. \tag{6}$$

The Hessian matrix for $F(x, y)$,

$$\frac{\psi''(x/y)}{y^3} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

is then positive semi-definite and thus F is convex. In particular, the level curve $F(x, y) = r$ bounds a convex set. Since $\psi'(0) > 0$ and ψ' is increasing, $\psi'(z) > 0$ for all $z > 0$. Indeed, since $\psi(0) > 0$, there exists for any $r > 0$, $y_0 > 0$ such that $F(0, y_0) = y_0\psi(0) = r$.

Letting $z := x/y$, we see that

$$F_1(x, y) = \psi'(z) \text{ and } F_2(x, y) = \psi(z) - z\psi'(z)$$

and therefore

$$-\frac{F_2(x,y)}{F_1(x,y)} = z - \frac{\psi(z)}{\psi'(z)} \tag{7}$$

where F_i is the derivative of F with respect to the i-th coordinate. By (6), this means that the slope of the tangent line to the curve $F(x,y) = r$ at (x,y) has negative slope and, along that curve, as x is increasing, the slope is decreasing. Thus, for $r > 0$, there exists $x_0 > 0$ such that the level curve $F(x,y) = r$ crosses the horizontal axis at x_0. Hence, the two axes and the level curve $F(x,y) = r$ enclose a bounded convex region.

Note that under our assumptions, $F(x,y) = 1$ is, in the first quadrant, the graph of a function with positive horizontal and vertical intercepts which is strictly decreasing and "concave down". We could have started with such a function f instead and note that $g(x) := f(x)/x$ is strictly decreasing and so we could define $F(x,y) := x/g^{-1}(y/x)$ for $x, y > 0$. It is easy to see that $F(x,y) = 1$ is equivalent to $y = f(x)$ and that $F(rx,ry) = rF(x,y)$.

It turns out that integrating across Pascal's triangle (with respect to arclength parameterization ds) satisfies

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim C\gamma^r$$

for a suitably defined C and γ.

Theorem 2. *For $F : \mathbb{R}^+ \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ such that the auxillary function $\psi(z) := F(z,1)$ is twice differentiable and satisfies $\psi(z) - z\psi'(z) \geq 0$ and $\psi''(z) \geq 0$, $\psi(0) > 0$, $\psi'(0) > 0$, let z_0 be the unique solution of*

$$\frac{\ln(1+z)}{\ln(1+1/z)} = \frac{\psi(z) - z\psi'(z)}{\psi'(z)},$$

$A := \psi'(z_0)$, $B := \psi(z_0) - z_0\psi'(z_0)$, and $\lambda := \ln(1+z_0)/B$. Then, as $r \to \infty$

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim \frac{z_0+1}{\psi(z_0)} \sqrt{\frac{A^2+B^2}{1+\lambda z_0(z_0+1)\psi''(z_0)}} \cdot e^{\lambda r}. \tag{8}$$

Proof. Let F_1 and F_2 denote partial derivatives with respect to the first and second variables respectively. Then, by (5),

$$F_1(x,y) = \psi'(x/y),$$
$$F_2(x,y) = \psi(x/y) - x\psi'(x/y)/y. \tag{9}$$

Let L be the length of the curve $F(x,y) = 1$ and we suppose that $x := x(s), y := y(s), s \in [0,L]$ is the arclength parametrization of this curve. Then $(rx(s), ry(s)), s \in [0,L]$ is a parametrization of the curve $F(x,y) = r$ and we have

$$\int_{F(x,y)=r} \binom{x+y}{x} ds = \int_0^L r\binom{rx+ry}{rx} ds. \tag{10}$$

Recall Stirling's formula [6, p. 253]:

$$\Gamma(x+1) = x^{x+\frac{1}{2}} e^{-x}\sqrt{2\pi} e^{\theta/12x}$$

for some $\theta \in (0,1)$ from which it follows that for some $\theta_1, \theta_2, \theta_3 \in (0,1)$, possibly depending on x and y,

$$\binom{x+y}{x} = \sqrt{\frac{x+y}{2\pi xy}} \cdot \frac{(x+y)^{x+y}}{x^x y^y} e^{(\theta_1/(x+y)-\theta_2/x-\theta_3/y)/12}$$

and thus, for $\theta_i \in [0,1]$ now depending on x, y, and r,

$$r\binom{rx+ry}{rx} = \sqrt{r \cdot \frac{x+y}{2\pi xy}} \cdot \left(\frac{(x+y)^{x+y}}{x^x y^y}\right)^r \cdot e^{(\theta_1/(x+y)-\theta_2/x-\theta_3/y)/12r}.$$

If we define, for $s \in (0, L)$,

$$\begin{aligned} g(s) &:= \sqrt{\frac{x+y}{2\pi xy}}, \\ h(s) &:= (x+y)\ln(x+y) - x\ln x - y\ln y, \text{ and} \\ E_r(s) &:= \exp(\theta_1/(x+y) - \theta_2/x - \theta_3/y)/12r), \end{aligned}$$

then

$$\int_{F(x,y)=r} \binom{x+y}{x} ds = \sqrt{r} \cdot \int_0^L g(s) e^{rh(s)} E_r(s) ds. \tag{11}$$

By the convexity of F, h attains its maximum value $h(\alpha)$ at a unique $\alpha \in (0, L)$ and so, by "Laplace's method" [2: p. 36 ff],

$$\int_0^L g(s) e^{rh(s)} ds \sim 2g(\alpha) e^{rh(\alpha)} \left[\frac{-\pi}{2rh''(\alpha)} \right]^{\frac{1}{2}} \tag{12}$$

(the first 2 in the right side of (12) is due to the fact that α is *not* an endpoint of [0,L]). Moreover, for r large, the contribution of the integrand when s is near 0 or L is negligible in the sense that (12) is independent of the choice of endpoints 0 and L and thus, for sufficiently small $\epsilon > 0$,

$$\int_\epsilon^{L-\epsilon} g(s) e^{rh(s)} ds \sim 2g(\alpha) e^{rh(\alpha)} \left[\frac{-\pi}{2rh''(\alpha)} \right]^{\frac{1}{2}}. \tag{13}$$

By the definition of $E_r(s)$, there exists k_1 (depending on ϵ) such that

$$\text{for any } s \in [\epsilon, L-\epsilon],\ 1 - \frac{k_1}{r} \le E_r(s). \tag{14a}$$

Note that although x and y can be near 0, $x + y$ is bounded away from 0 and so there exists k_2 such that

$$\text{for any } s \in [0, L],\ E_r(s) \le 1 + \frac{k_2}{r}. \tag{14b}$$

Letting $A(\epsilon) := [0, \epsilon] \cup [L - \epsilon, L]$,

$$\frac{\int_{F(x,y)=r} \binom{x+y}{x} ds}{\sqrt{r} \int_0^L g(s) e^{rh(s)} ds} = \frac{\int_\epsilon^{L-\epsilon} g(s) e^{rh(s)} E_r(s) ds}{\int_0^L g(s) e^{rh(s)} ds} + \frac{\int_{A(\epsilon)} g(s) e^{rh(s)} E_r(s) ds}{\int_0^L g(s) e^{rh(s)} ds}.$$

By (12), (13), and (14), the first summand converges to 1 and the second converges to 0. Hence,

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim \sqrt{r} \cdot \int_0^L \sqrt{\frac{x+y}{2\pi xy}} \cdot \left(\frac{(x+y)^{x+y}}{x^x y^y} \right)^r ds. \tag{15}$$

Combining (12) and (15) yields

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim \sqrt{-\frac{x(\alpha) + y(\alpha)}{h''(\alpha) x(\alpha) y(\alpha)}} \cdot e^{cr} \tag{16}$$

where c maximizes $(x + y)\ln(x + y) - x \ln x - y \ln y$ subject to the constraint $F(x, y) = 1$. By Lagrangian multipliers, a solution satisfies

$$\begin{cases} \ln(x+y) - \ln x = \lambda F_1(x,y) \\ \ln(x+y) - \ln y = \lambda F_2(x,y). \end{cases} \tag{17}$$

By (5), $F_1(x, y) = \psi'(x/y)$ and $F_2(x, y) = \psi(x/y) - x\psi'(x/y)/y$. Letting $z := x/y$, (17) implies

$$\frac{\ln(1+z)}{\ln(1+1/z)} = \frac{\psi(z) - z\psi'(z)}{\psi'(z)}. \tag{18}$$

Since the left side is increasing and the right side decreasing (by (6)), there is a unique solution z_0 of (18). There is then, using (5) and the assumption that $F(x_0, y_0) = 1$, a unique solution to (17), namely

$$x_0 := \frac{z_0}{\psi(z_0)},\ y_0 := \frac{1}{\psi(z_0)},\ \text{and } \lambda := \frac{\ln(1+1/z_0)}{\psi'(z_0)}. \tag{19}$$

Let x_0', y_0', x_0'', and y_0'' be the corresponding values of x', y', x'', and y'' at the α satisfying $z_0 = z(\alpha)$. With

$$A := F_1(x_0, y_0) = \psi'(z_0), B := F_2(x_0, y_0) = \psi(z_0) - z_0\psi'(z_0),$$

since x and y are related by $F(x, y) = 1$, differentiating gives

$$F_1(x, y)x' + F_2(x, y)y' = 0 \tag{20}$$

which when evaluating at α gives

$$Ax_0' = By_0' = 0. \tag{21}$$

Since x, y is an arc-length parameterization, $(x_0')^2 + (y_0')^2 = 1$ and thus

$$x_0' = \frac{\pm B}{\sqrt{A^2+B^2}} \text{ and } y_0' = \frac{\mp A}{\sqrt{A^2+B^2}}.$$

Differentiating (20) and evaluating at α yields

$$\begin{aligned} 0 = {} & F_{11}(x_0, y_0)(x_0')^2 + 2F_{12}(x_0, y_0)x_0'y_0' + F_{22}(x_0, y_0)(y_0')^2 \\ & + F_1(x_0, y_0)x_0'' + F_2(x_0, y_0)y_0''. \end{aligned} \tag{22}$$

Using $F(x, y) = y\psi(x/y)$, $F(x, y) = 1$ and $z = x/y$, it follows that

$$F_{11}(x, y) = \psi(z)\psi''(z), F_{12}(x, y) = z\psi(z)\psi''(z), F_{22}(x, y) = z^2\psi(z)\psi''(z). \tag{23}$$

Rewriting (23), we have

$$0 = \frac{(Az_0 + B)^2}{A^2 + B^2} + Ax_0'' + By_0''.$$

Differentiating $(x')^2 + (y')^2 = 1$ and using (21), $Bx_0'' = Ay_0''$ and thus

$$\begin{aligned} x_0'' &= -\frac{A}{(A^2+B^2)^2}\psi(z_0)\psi''(z_0)(Az_0 + B)^2, \\ y_0'' &= -\frac{B}{(A^2+B^2)^2}\psi(z_0)\psi''(z_0)(Az_0 + B)^2. \end{aligned} \tag{24}$$

To find $h''(\alpha)$, we twice differentiate $h = (x + y)\ln(x + y) - x\ln x - y\ln y$ to get

$$h'' = (x'' + y'')\ln(x + y) - x''\ln x - y''\ln y + \frac{(x' + y')^2}{x + y} - \frac{(x')^2}{x} - \frac{(y')^2}{y}$$

and so

$$h''(\alpha) = x_0''\ln(1 + 1/z_0) + y_0''\ln(1 + z_0) - \frac{(x_0'y_0 - x_0y_0')^2}{x_0y_0(x_0 + y_0)}.$$

Using (24), it follows that

$$h''(\alpha) = -\frac{\psi(z_0)^3}{A^2 + B^2}\left(\lambda\psi''(z_0) + \frac{1}{z_0(z_0 + 1)}\right)$$

and equation (16) can be rewritten to give the result. □

Remarks. Note that

$$\frac{x + y}{x} + \frac{x + y}{y} = \frac{(x + y)^2}{xy}$$

and so

$$e^{\ln(x_0+y_0)-\ln x_0+\ln(x_0+y_0)-\ln y_0} = e^{\ln(x_0+y_0)-\ln x_0} + e^{\ln(x_0+y_0)-\ln y_0}$$

which, by (17), can be rewritten as

$$e^{\lambda A+\lambda B} = e^{\lambda A} + e^{\lambda B} \tag{25}$$

where $A := \psi'(z_0) = F_1(x_0, y_0)$ and $B := \psi(z_0) - z_0\psi'(z_0) = F_2(x_0, y_0)$. Note that $Ax_0 + By_0 = 1$ so

$$\lambda = \lambda A x_0 + \lambda B y_0 = (x_0 + y_0)\ln(x_0 + y_0) - x_0 \ln x_0 - y_0 \ln y_0 = h(\alpha).$$

Thus we can characterize λ as e^γ where γ is the unique positive solution of $\gamma^{A+B} = \gamma^A + \gamma^B$.

If $F(kx, ky) = k^\alpha F(x, y)$ (i.e., F is homogeneous of degree α), then $G(x, y) := F(x, y)^{1/\alpha}$ is homogeneous of degree 1 and theorem 2 yields a result of the form

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim C\gamma^{r^{1/\alpha}}.$$

It is curious that the right side of (18) is negative of $z - \psi(z)/\psi'(z)$, the function whose iteration is 'Newton's method' for ψ.

Example 1. Let $F(x, y) = ax + by$, $a, b > 0$. Then $\psi(z) = az + b$. No matter what z_0 is, $A := \psi'(z_0) = a$ and $B := \psi(z_0) - z_0\psi'(z_0) = az_0 + b - z_0 a = b$. Let γ be the unique positive solution of $\gamma^{a+b} = \gamma^a + \gamma^b$. Since $e^\lambda = \gamma$ and $\lambda b = \ln(1 + z_0)$, $z_0 = \gamma^b - 1$. Since $\psi'' = 0$, the constant term on the right side of (8) is

$$C = \sqrt{A^2 + B^2}(z_0 + 1)/\psi(z_0) = \sqrt{a^2 + b^2}\gamma^b/(a(\gamma^b - 1) + b)$$

and thus

$$\int_{ax+by=r} \binom{x+y}{x} ds \sim \sqrt{a^2 + b^2}\,\frac{\gamma^{a+b}}{b\gamma^a + a\gamma^b}\gamma^r.$$

Note the similarity to (2).

Example 2. Let $F(x, y) = \sqrt{x^2 + y^2}$. The $\psi(z) = \sqrt{z^2 + 1}$, $\psi'(z) = z/\sqrt{z^2 + 1}$, and $\psi''(z) = 1/(z^2 + 1)^{3/2}$. Note that $z_0 = 1$ since 1 is the (unique) solution of

$$\frac{\ln(1 + z)}{\ln(1 + 1/z)} = \frac{\psi(z) - z\psi'(z)}{\psi'(z)} = \frac{1}{z}.$$

Then $A = \psi'(1) = 1/\sqrt{2}$, $B = \psi(1) - \psi'(1) = 1/\sqrt{2}$, and $\lambda = \ln(1 + 1)/\psi'(1) = \sqrt{2}\ln 2$. It follows that

$$\int_{\sqrt{x^2+y^2}=r} \binom{x+y}{x} ds \sim \sqrt{\frac{2}{1 + \ln 2}} \cdot (2^{\sqrt{2}})^r.$$

Example 3. Let $F(x, y) = (x^p + y^p)^{1/p}$ where $p > 1$. The method of example 2 can be followed. We get

$$\int_{(x^p+y^p)^{1/p}=r} \binom{x+y}{x} ds \sim \sqrt{\frac{2}{1 + (p - 1)\ln 2}} \cdot (2^{2^{1-1/p}})^r.$$

Examples 2 and 3 are special cases where F is symmetric and thus $z_0 = 1$. We state the generalization as a corollary of Theorem 2.

Corollary 1. *For F satisfying the hypotheses of Theorem 2 as well as $F(x, y) = F(y, x)$ for all x, y,*

$$\int_{F(x,y)=r} \binom{x+y}{x} ds \sim \sqrt{\frac{2\psi'(1)}{\psi'(1) + 2\psi''(1)\ln 2}} \cdot 2^{r/\psi'(1)}$$

as $r \to \infty$.

Proof. By symmetry, $\psi(z) = z\psi(1/z)$. Differentiating, get $\psi'(z) = \psi(1/z) - \psi'(1/z)/z$. Hence

$$\frac{\ln(1+1)}{\ln(1+1/1)} = 1 = \frac{\psi(1) - 1\cdot\psi'(1)}{\psi'(1)}$$

and so $z_0 = 1$. Then $A = B = \psi'(1)$, $\psi(1) = 2\psi'(1)$, and the result follows. □

4. DISCRETE CASE REVISITED.

For homogeneous (of degree 1) F, and any function G, it is natural to define, where m, n denote integers,

$$\limsup_{F(m,n)\to\infty} G(m,n) := \limsup_{R\to\infty} \sup_{F(m,n)=R} G(m,n) = \lim_{R\to\infty} \sup_{F(m,n)>R} G(m,n).$$

It follows immediately from Theorem 1 that for $F(x,y) = rx + sy$ $(r, s > 0)$, and λ as defined in Theorem 2,

$$\limsup_{F(m,n)\to\infty} \frac{\ln\binom{m+n}{m}}{F(m,n)} = \lambda$$

or, equivalently,

$$\limsup_{F(m,n)\to\infty} \binom{m+n}{m}^{1/F(m,n)} = \gamma.$$

This equation generalizes:

Theorem 3. *For* $F : \mathbb{R}^+ \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ *such that the auxillary function* $\psi(z) := F(z,1)$ *is twice differentiable and satisfies* $\psi(z) - z\psi'(z) \geq 0$ *and* $\psi''(z) \geq 0$, $\psi(0) > 0$, $\psi'(0) > 0$, *let* z_0 *be the unique solution of*

$$\frac{\ln(1+z)}{\ln(1+1/z)} = \frac{\psi(z) - z\psi'(z)}{\psi'(z)},$$

and let $\gamma := (1+1/z_0)^{1/\psi'(z_0)}$. *Then*

$$\limsup_{F(m,n)\to\infty} \binom{m+n}{m}^{1/F(m,n)} = \gamma. \tag{26}$$

Proof. From the previous section,

$$\gamma = e^{\lambda} = \sup_{F(x,y)=1} \frac{(x+y)^{x+y}}{x^x y^y}$$

and so, by (10),

$$\binom{Rx+Ry}{Rx}^{1/R} = \left(\sqrt{\frac{x+y}{2\pi Rxy}}\right)^{1/R} \left(1 + O\left(\frac{1}{R}\right)\right) \frac{(x+y)^{x+y}}{x^x y^y} \tag{27}$$

and thus

$$\gamma = \lim_{R\to\infty} \sup_{F(x,y)=1} \binom{Rx+Ry}{Rx}^{1/R} = \limsup_{R\to\infty} \sup_{F(x,y)=R} \binom{x+y}{x}^{1/F(x,y)} =$$
$$\lim_{R\to\infty} \sup_{F(x,y)\geq R} \binom{x+y}{x}^{1/F(x,y)} = \limsup_{F(x,y)\to\infty} \binom{x+y}{x}^{1/F(x,y)}.$$

Given x and y, let $m := \lceil x \rceil$ and $n := \lceil y \rceil$. By the convexity of F, the level curve $F(x,y) = 1$ is contained in a proper annulus centered at the origin and so $F(m,n) - F(x,y)$ is bounded (as a function of x and y) and $\{\binom{x+y}{x} : F(x,y) = 1\}$ is bounded. The last fact implies, by (10), that $\binom{x+y}{x}^{1/F(x,y)}$ is bounded.

Note that since the curve $F(x,y) = 1$ is the graph of a decreasing function, $F(x,y) \le F(m,n)$. Suppose now that $F(x,y) \ge R$. By the mean value theorem, for some $C \in (F(x,y), F(m,n))$ and some M,

$$\begin{aligned}
&\binom{x+y}{x}^{1/F(x,y)} - \binom{m+n}{m}^{1/F(m,n)} \\
&= \ln\binom{x+y}{x} \cdot \binom{x+y}{x}^{1/C} \cdot \left(-\frac{1}{C^2}\right) \cdot (F(x,y) - F(m,n)) \\
&\le \frac{1}{R}\left(\frac{1}{F(x,y)} \ln\binom{x+y}{x}\right) \cdot \binom{x+y}{x}^{1/F(x,y)} \cdot (F(m,n) - F(x,y)) \\
&\le \frac{M}{R}\binom{x+y}{x}^{1/F(x,y)}
\end{aligned}$$

for some M. Since the level curve $\binom{x+y}{x} = c$ is the graph of a decreasing function,

$$\binom{x+y}{x} \le \binom{m+n}{m}.$$

Then,

$$\binom{x+y}{x}^{1/F(x,y)} \cdot (1 - M/R) \le \binom{x+y}{x}^{1/F(m,n)} \le \binom{m+n}{m}^{1/F(m,n)}$$

and so

$$\limsup_{F(x,y)\to\infty} \binom{x+y}{x}^{1/F(x,y)} \le \limsup_{F(m,n)\to\infty} \binom{m+n}{m}^{1/F(m,n)}.$$

The reverse inequality is trivial and the result follows. □

Example 4. When $F(x,y) := \sqrt{x^2+y^2}$, we have, using example 2,

$$\limsup_{\sqrt{m^2+n^2}\to\infty} \binom{m+n}{m}^{1/\sqrt{m^2+n^2}} = 2^{\sqrt{2}}.$$

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