UNIVERSITY AT STONY BROOK

CEAS Technical Report 788

HYPERREAL TRANSIENTS IN TRANSFINITE RLC NETWORKS

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January 12, 2001

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Abstract — Up to the present time, there have been no transient analyses of RLC transfinite networks. Standard analyses of transfinite networks have been restricted to purely resistive ones. In this paper, it is shown how nonstandard analysis can be used to examine the transient behavior of transfinite networks having lumped resistors, inductors, and capacitors. To do so, the time line is expanded into the hyperreal time line, and the transients obtained take on hyperreal values. It is also shown how the diffusion of signals on artificial RC cables and the propagation of waves on artificial RLC transmission lines can "pass through infinity" and penetrate transfinite extensions of those cables and lines. Less precisely but more suggestively, we can say that diffusions and waves can reach—with appreciable values—nodes that are transfinitely far away from their starting points, but that it will take infinitely long times in order to get there.

KEY WORDS: hyperreal transients; hyperreal time line; transfinite RLC networks; nonstandard analysis

1 Introduction

All the analyses so far of transfinite electrical networks have been restricted to the "resistive case," that is, to the case where the elements are (linear or nonlinear) resistances and also independent voltage and current sources (see [9] or [11] and the bibliographies therein). Moreover, except for two recent papers [12], [13], those analyses were standard—employing only the real numbers; as a result, a variety of restrictions had to be imposed on such networks in order to ensure the existence of voltage-current regimes. These difficulties can be completely overcome by using instead nonstandard analysis. Indeed, by allowing hyperreal voltages and currents, all restrictions on the transfinite networks can be removed except for the property of restorability, which avoids a certain kind of graphical pathology (see [12, Theorem 5.3] or [13, Theorem 4.3]).

The purpose of this paper is to show that nonstandard analysis also allows us to introduce reactive elements into transfinite networks and to establish transient behavior, wherein voltages and currents are hyperreals depending upon hyperreal time. For instance, we show in Sec. 8 that an artificial wave produced by a unit step of voltage applied to the input of a lumped RLC transmission line can "pass through infinity" and continue on through transfinite extensions of that line during unlimited hyperreal times.

To understand this paper, some knowledge of nonstandard analysis and of transfinite graphs is needed. In the following, we will provide enough definitions and explanations to make our use of nonstandard analysis comprehensible. For tutorials on this subject, one might refer to [3] or [4] or to the substantial textbook [2]. There is some divergence in terminology for nonstandard analysis; we follow that used in [2][1]

As for transfinite graphs and networks, one may refer to either of the books [9] or [11] or to the tutorial/survey article [10].

2 Some Elements of Nonstandard Analysis

In this section we point out some definitions and results from nonstandard analysis without proving anything. The basic idea is the following: Let $\boldsymbol{R}$ and $\boldsymbol{R}_+$ denote respectively the real line $-\infty < x < \infty$ and the nonnegative real line $0 \leq x < \infty$. These can be expanded into the "hyperreal line" ${}^*\boldsymbol{R}$ and the "nonnegative hyperreal line" ${}^*\boldsymbol{R}_+$ by using equivalence classes of sequences of real numbers as determined by a chosen "nonprincipal ultrafilter" $\mathcal{F}$. Each such equivalence class is a "hyperreal number." It is convenient to refer to real numbers simply as "reals" and to hyperreal numbers as "hyperreals." We will use boldface notation for hyperreals in order to distinguish them from reals.

More specifically, let $\boldsymbol{N} = \{0, 1, 2, \ldots\}$ be the set of all the natural numbers. A *non-principal ultrafilter $\mathcal{F}$ on $\boldsymbol{N}$* is a collection of nonempty subsets of $\boldsymbol{N}$ satisfying the following

[1]Except for the notation of a hyperreal: We write $\langle i_n \rangle$ as in [4], whereas $[i_n]$ is used in [2].

axioms:

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

2. If $A \in \mathcal{F}$ and $A \subset B \subset A$, then $B \in \mathcal{F}$.

3. For any $A \subset \boldsymbol{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ but not both. Here, $A^c = \boldsymbol{N} \backslash A$ denotes the complement of A in $\boldsymbol{N}$.

4. No finite subset of $\mathcal{N}$ is a member of $\mathcal{F}$.

As a result of these assumptions, we have the following properties:

a. $\emptyset \notin \mathcal{F}$ and $\boldsymbol{N} \in \mathcal{F}$, where $\emptyset$ denotes the emptyset.

b. If $\{A_1, A_2, \ldots, A_k\}$ is a finite collection of mutually disjoint subsets of $\mathcal{F}$, then no more than one of them is a member of $\mathcal{F}$. If in addition $\cup_{j=1}^{k} A_j = \boldsymbol{N}$, then exactly one of the A_j is a member of $\mathcal{F}$.

c. Every cofinite set (i.e., the complement of a finite set) in $\boldsymbol{N}$ is a member of $\mathcal{F}$.

d. $\mathcal{F}$ is a maximal filter in the following sense: A *proper filter* on $\boldsymbol{N}$ is by definition a collection $\mathcal{G}$ of subsets of $\boldsymbol{N}$ with $\emptyset \notin \mathcal{G}$ and satisfying Conditions 1 and 2 above. There is no proper filter that is larger than $\mathcal{F}$ in the sense that $\mathcal{F}$ is a proper subset of $\mathcal{G}$. On the other hand, for each proper filter $\mathcal{G}$, there is an ultrafilter $\mathcal{F}$ having $\mathcal{G}$ as a subset (possibly, $\mathcal{G} = \mathcal{F}$).

There are many nonprincipal ultrafilters on $\boldsymbol{N}$. Let us choose and fix our attention on one of them, say, $\mathcal{F}$. Also, let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two sequences of real numbers. We call these sequences *equivalent modulo* $\mathcal{F}$—or simply *equivalent* when a chosen and fixed $\mathcal{F}$ is understood—if $\{n \in \boldsymbol{N} : x_n = y_n\} \in \mathcal{F}$. This is truly an equivalence relation. As a result, the set of all sequences of reals is partitioned into equivalence classes, each of which is defined to be a *hyperreal.* Each member of an equivalence class is a *representative* of that hyperreal, and that hyperreal is denoted by $\mathbf{x} = \langle x_n \rangle$ or $\mathbf{x} = \langle x_1, x_2, x_3, \ldots \rangle$, where $\{x_n\}_{n=0}^{\infty}$ is any such representative. Every real $x \in \boldsymbol{R}$ has a hyperreal version $\langle x, x, x, \ldots \rangle$.

In this way, we view $\boldsymbol{R}$ as being a subset of the set $^*\boldsymbol{R}$ of all hyperreals, in which case it is convenient to use the same symbol for the real and the hyperreal. For example, 2 is a real, and 2 also denotes the corresponding hyperreal $\langle 2, 2, 2, \ldots \rangle$.

If a condition depending upon n holds for all n in some set $F \in \mathcal{F}$, we will simply say that it holds "for almost all n" or simply "a.e.".[2] For example, the hyperreals $\mathbf{x} = \langle x_n \rangle$ and $\mathbf{y} = \langle y_n \rangle$ are defined to be *equal* (i.e., $\mathbf{x} = \mathbf{y}$) if $\{n \in \boldsymbol{N} : x_n = y_n\} = F \in \mathcal{F}$, and we say in this case that $x_n = y_n$ a.e. Furthermore, addition, multiplication, inequality, and absolute value are defined componentwise on the representatives of hyperreals. That is, if $\mathbf{x} = \langle x_n \rangle$ and $\mathbf{y} = \langle y_n \rangle$, then $\mathbf{x} + \mathbf{y} = \langle x_n + y_n \rangle$ and $\mathbf{xy} = \langle x_n y_n \rangle$. Also, $\mathbf{x} < \mathbf{y}$ means $x_n < y_n$ a.e., and $\mathbf{x} \leq \mathbf{y}$ is defined similarly. Furthermore, $|\mathbf{x}| = \langle |x_n| \rangle$. Finally, $^*\boldsymbol{R}_+$ will denote the set of all nonnegative hyperreals: $\mathbf{x} = \langle x_n \rangle \in^* \boldsymbol{R}_+$ if and only if $x_n \geq 0$ a.e.

The hyperreal $\langle x_n \rangle$ is called *infinitesimal* if, for every positive real ϵ, we have $\{n \in \boldsymbol{N} : |x_n| < \epsilon\} \in \mathcal{F}$, that is, if $|x_n| < \epsilon$ a.e. Also, $\langle x_n \rangle$ is called *unlimited* if $|x_n| > \epsilon$ a.e. for every positive real ϵ. Thus, the reciprocal $\langle x_n^{-1} \rangle$ of an infinitesimal $\langle x_n \rangle$ is unlimited, and conversely. A *limited* hyperreal is one that is not unlimited. Thus, $\mathbf{x} = \langle x_n \rangle$ is limited if and only if there is a $\gamma \in \boldsymbol{R}_+$ such that $|x_n| < \gamma$ a.e. A hyperreal that is neither infinitesimal nor unlimited is called *appreciable*. Thus, $\langle x_n \rangle$ is appreciable if, for every ϵ and γ with $0 < \epsilon < \gamma < \infty$, we have that $\epsilon < x_n < \gamma$ a.e. Around each real $\mathbf{x} = \langle x, x, x, \ldots \rangle$ in $^*\boldsymbol{R}$, there is a set of hyperreals $\mathbf{y} = \langle y_1, y_2, y_3, \ldots \rangle$ that are infinitesimally close to $\mathbf{x}$ (i.e., $|\mathbf{x} - \mathbf{y}|$ is infinitesimal for each such $\mathbf{y}$). The set of such hyperreals is called the *halo* of $\mathbf{x}$, and $\mathbf{x}$ is called the *shadow* or *standard part* of every $\mathbf{y}$ in that halo.

Since every cofinite set is a member of every nonprincipal ultrafilter, any of the adjectives: infinitesimal, appreciable, limited, and unlimited holds for $\mathbf{x} = \langle x_n \rangle$ whenever the corresponding inequality on x_n holds for all n in a cofinite subset of $\boldsymbol{N}$. Moreover, we are free to change the values of x_n in $\mathbf{x} = \langle x_n \rangle$ for all n in any subset of $\boldsymbol{N}$ not in $\mathcal{F}$; this will not alter $\mathbf{x}$.

[2] The abbreviation "a.e." stands for "almost everywhere". Although brief and convenient, "a.e." is rather a misnomer, for the set of those n for which the condition holds can be a very small subset of $\boldsymbol{N}$.

3 Hyperreal Transients on the Hyperreal Time Line

When discussing hyperreal transients, we will take the *hyperreal time line* to be ${}^*\boldsymbol{R}_+$, and $\mathbf{t} = \langle t_n \rangle \in {}^*\boldsymbol{R}_+$ will be *hyperreal time.* Thus, $\mathbf{t} = \langle n \rangle$ is an example of an unlimited point in hyperreal time. Arbitrarily large unlimited time points exist in ${}^*\boldsymbol{R}_+$. Thus, we can view the hyperreal time line as starting at 0, passing through the infinitesimals, then through the appreciable hyperreals (whose shadows comprise the conventional time line), and finally through the unlimited hyperreals (infinite values of time). Any unlimited time point is an upper bound on all the appreciable time points, but there is no upper bound on the set of unlimited time points.

Both ${}^*\boldsymbol{R}$ and ${}^*\boldsymbol{R}_+$ are "internal sets." More generally, if A_n is a subset of $\boldsymbol{R}$ for each n (possibly, $A_n = \boldsymbol{R}$), then the subset $\langle A_n \rangle$ of ${}^*\boldsymbol{R}$, defined by

$$\langle x_n \rangle \in {}^*\boldsymbol{R} \text{ if and only if } \{n \in \boldsymbol{N} : x_n \in A_n\} \in \mathcal{F},$$

is called an "internal set"; it is a subset of ${}^*\boldsymbol{R}$ [2, page 126].

Next, let $\{f_n\}_{n=0}^{\infty}$ be a sequence of standard (i.e., conventional, real-valued) functions mapping $\boldsymbol{R}_+$ into $\boldsymbol{R}$. Then, the ${}^*\boldsymbol{R}_+$-valued function $\mathbf{f} = \langle f_n \rangle$ can be defined on ${}^*\boldsymbol{R}_+$ by setting

$$\mathbf{f}(\mathbf{t}) := \langle f_n(t_n) \rangle, \quad \mathbf{t} = \langle t_n \rangle \in^* \boldsymbol{R}_+.$$

Similarly, another sequence $\{g_n\}_{n=0}^{\infty}$ of standard functions mapping $\boldsymbol{R}_+$ into $\boldsymbol{R}$ is taken to be *equivalent to* $\{f_n\}_{n=0}^{\infty}$ if $g_n = f_n$ a.e. (i.e., $\{n \in \boldsymbol{N} : f_n = g_n\} \in \mathcal{F}$). As before, $\mathbf{f} = \langle f_n \rangle$ denotes the equivalence class of all such equivalent functions. We will view such an $\mathbf{f}$ as a *hyperreal transient* defined on the hyperreal time line.

More generally, if each f_n has a domain $A_n \subset \boldsymbol{R}$, then $\langle f_n \rangle$ is an "internal function" $\mathbf{f}$ mapping the internal set $\langle A_n \rangle \subset {}^*\boldsymbol{R}$ into ${}^*\boldsymbol{R}$; it is defined by $\langle f_n \rangle(\langle x_n \rangle) = \langle f_n(x_n) \rangle$ [2, page147].

As an example, let

$$f_0(t) = 1, \;\; f_n(t) = 1 - e^{-t/n}, \;\; t \in \boldsymbol{R}_+, \;\; n = 1, 2, 3, \ldots$$

For $n > 0$, each f_n is a strictly increasing function with $f_n(0) = 0$ and $\lim_{t \to \infty} f_n(t) = 1$. Then, for $\mathbf{f} = \langle f_n \rangle$, we have $\mathbf{f}(\mathbf{t}) = \langle 1 - e^{-t_n/n} \rangle$. The hyperreal function $\mathbf{f}$ is strictly increasing; indeed, if $\mathbf{t} = \langle t_n \rangle < \mathbf{x} = \langle x_n \rangle$, we have $t_n < x_n$ a.e. and $1 - e^{-t_n/n} < 1 - e^{-x_n/n}$ a.e., and thus $\mathbf{f}(\mathbf{t}) < \mathbf{f}(\mathbf{x})$. Furthermore, $\mathbf{f}(\mathbf{t})$ is a positive infinitesimal when $\mathbf{t} = \langle t_n \rangle$ is positive and limited; indeed, $0 < t_n < T$ a.e. for some fixed real $T \in \mathbb{R}_+$, and therefore

$$0 < f_n(t_n) < 1 - e^{-T/n} \to 0, \ n \to \infty.$$

Thus, on the limited hyperreal time line, $\mathbf{f}(\mathbf{t})$ increases but remains infinitesimal. However, on the unlimited hyperreal time line, $\mathbf{f}(\mathbf{t})$ continues to increase and eventually increases through limited hyperreals but remains less than 1; indeed, we can choose t_n such that t_n/n approaches any positive real x so that $1 - e^{-t_n/n}$ approaches $1 - e^{x}$.

4 Nonstandard Versions of Restorable Transfinite Graphs

For finite RLC networks, our use of nonstandard analysis is not needed; standard analysis certainly serves well enough. But, for many conventionally infinite RLC networks and for all transfinite RLC networks, nonstandard analysis provides for the first time a means of establishing and examining transient behavior. However, certain pathological transfinite networks do not (as yet) have nonstandard versions; but, the "restorable ones" do. Let us briefly explain what "restorability" means, and refer the reader to [12, Sec. 5] or to [13, Sec. 4] for a thorough discussion.

Let $\mathcal{G}^\nu$ be a transfinite graph of rank ν having countably many branches.[3] We can "open" a branch simply by removing it from $\mathcal{G}^\nu$, and we can "short" it by removing it and then coalescing its two nodes into a single node. This works even for transfinite nodes. Furthermore, we can render $\mathcal{G}^\nu$ into a finite graph $\mathcal{G}_n$ as follows: Each transfinite node n^γ is a set of extremities (called "tips") of transfinite paths that reach that node. We can short portions of such paths to eliminate all such tips in n^γ, rendering n^γ into a conventional node. We can then open enough branches to obtain the finite graph $\mathcal{G}_n$. Moreover, we can repeatedly generate in this way an expanding sequence $\{\mathcal{G}_n\}_{n=0}^{\infty}$ of finite graphs such that $\mathcal{G}_n$ is a subgraph of $\mathcal{G}_{n+1}$ for each n. We can get from $\mathcal{G}_n$ to $\mathcal{G}_{n+1}$ by restoring finitely many branches. If it is possible to choose $\{\mathcal{G}_n\}_{n=0}^{\infty}$ such that the restoration of all the branches

[3] See either of [9, Chap. 2] or [11, Chap. 2] for a discussion of transfinite graphs.

produces the same connections between tips as those in $\mathcal{G}^\nu$, we say that $\mathcal{G}^\nu$ is "restorable." Necessary and sufficient conditions for $\mathcal{G}^\nu$ to be restorable are given in [12, Theorem 5.3] and [13, Theorem 4.3].

Actually, many such sequences can be generated depending upon how we open, short, and restore branches. Two such sequences $\{\mathcal{G}_n\}_{n=0}^\infty$ and $\{\mathcal{H}_n\}_{n=0}^\infty$ will be called *equivalent* if $\mathcal{G}_n = \mathcal{H}_n$ a.e.[4] In this way, we partition all such sequences into equivalence classes (modulo $\mathcal{F}$), and each such class is taken to be a nonstandard version $\mathcal{G}^\nu_{\rm ns}$ of $\mathcal{G}^\nu$. $\mathcal{G}^\nu$ will have many such nonstandard versions.

As an example, consider the transfinite graph of Fig. 1(a) consisting of two one-way-infinite grounded ladders, with the first ladder connected at its infinite extremity to the input of the second ladder. That second ladder does not have anything connected to its infinite extremity. The small circles denote transfinite nodes of rank 1. The bottom ground line is an ordinary node of infinite degree.[5]

To obtain a finite graph, we short and open branches as indicated in Fig. 1(b). The shorted branches are indicated by heavy lines, and the opened branches by gaps. The other branches are indicate as resistors, but later on we will allow them to be RLC one-ports. This leaves a finite graph, say, $\mathcal{G}_0$. We can generate a sequence of finite graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ by restoring the shorted and opened branches one at a time, proceeding from left to right and alternating between the first and second ladders. Upon completion of this infinite process, we will have restored the transfinite graph of Fig. 1(a). The sequence $\{\mathcal{G}_n\}_{n=0}^\infty$ is a representative of an equivalence class $\mathcal{G}^\nu_{\rm ns} = \langle \mathcal{G}_n \rangle$ (modulo a given $\mathcal{F}$) of sequences of finite graphs. $\mathcal{G}^\nu_{\rm ns}$ is a nonstandard version of $\mathcal{G}^\nu$. Other nonstandard versions can be obtained by restoring branches two at a time, or three at a time, and so forth—or by varying the way the branches are initially opened and shorted.

[4] Here, equality means that $\mathcal{G}_n$ and $\mathcal{H}_n$ are graphically isomorphic.

[5] At this point, we are using the resistor symbol merely to distinguish a branch from a short; the solid lines along the bottom path are shorts that comprise the said ordinary node.

5 Nonstandard RLC Networks

Starting with a restorable transfinite graph $\mathcal{G}^\nu$ of rank ν, we can construct a nonstandard graph $\mathcal{G}^\nu_{\mathbf{ns}}$ as stated above. We can then obtain a *nonstandard transfinite RLC network* $\mathbf{N}^\nu_{\mathbf{ns}}$ by assigning resistors, inductors, capacitors, and independent sources to the branches of $\mathcal{G}^\nu_{\mathbf{ns}}$. (In this paper, we allow neither dependent sources nor mutual coupling between branches.) Thus, $\mathbf{N}^\nu_{\mathbf{ns}}$ is an equivalence class $\langle \mathrm{N}^\nu_n \rangle$ (modulo $\mathcal{F}$) of sequences of finite RLC networks, where $\{\mathrm{N}^\nu_n\}_{n=0}^\infty$ is a representative sequence of finite RLC networks corresponding to the representative sequence $\{\mathcal{G}_n\}_{n=0}^\infty$ for the nonstandard graph $\mathcal{G}^\nu_{\mathrm{ns}}$ of $\mathrm{N}^\nu_{\mathrm{ns}}$. It will be understood that the electrical parameters have fixed values independent of n.[6]

Then, with the sources being functions of time $t \in \mathbb{R}_+$ and with initial conditions assigned to the inductor currents and capacitor voltages, we obtain a transient regime of real values in each N_n. For instance, for each branch b, we have a sequence $\{i_{b,n}(t)\}_{n=0}^\infty$ of branch currents, where $i_{b,n}(t)$ is the current in b as a branch in N_n. Thus, for each fixed real $t \in \mathbb{R}_+$, $\langle i_{b,n}(t)\rangle$ is a hyperreal current in b with $\{i_{b,n}(t)\}_{n=0}^\infty$ being a representative sequence for it. (If b has not yet been restored in N_n, we can assign any value to $i_{b,n}(t)$ without affecting the hyperreal current for b.)

More generally, we can take time to be hyperreal as well: $\mathbf{t} = \langle t_n \rangle \in {}^*\mathbb{R}_+$. Then, $\langle i_{b,n}(t_n)\rangle \in {}^*\mathbb{R}_+$ is a hyperreal current for the chosen $\mathbf{t} \in {}^*\mathbb{R}_+$; it is defined even for unlimited (i.e., infinitely large) $\mathbf{t}$. More particularly, ${}^*\mathbb{R}_+ = \langle A_n \rangle$ is an internal subset of ${}^*\mathbb{R}$, wherein $A_n = \mathbb{R}_+$ for each n [2, page 126], and $\mathrm{i}_b = \langle i_{b,n}\rangle$ is an internal function [2, page 147] mapping $\mathbf{t} = \langle t_n \rangle \in {}^*\mathbb{R}_+$ into $\mathrm{i}(\mathbf{t}) = \langle i_{b,n}(t_n)\rangle \in {}^*\mathbb{R}$. Hence, $\mathrm{i}_b(\mathbf{t})$, where $\mathbf{t} \in^* \mathbb{R}_+$, is well-defined. In fact, $\mathbf{i}_b$ is an internal function defined on any internal subset of ${}^*\mathbb{R}_+$, such as the set $A = \langle [0, \epsilon_n]\rangle$, where ϵ_n is any real number for each n. In this way, that branch current $\mathbf{i}_b$ is a *nonstandard transient* mapping the hyperreal time line ${}^*\mathbb{R}_+$ into the set ${}^*\mathbb{R}$ of hyperreals.

In the same way, we have a nonstandard transient for each branch voltage and for each node voltage with respect to a chosen ground node. The set of all these nonstandard

[6] **Still more generality can be obtained by allowing those parameters to be hyperreals (i.e., by letting the parameter values vary with n). We will not pursue that possibility in this paper.**

transients is a *nonstandard transient regime* for N^{ν}_{ns}.

In the rest of this paper, we shall illustrate these ideas by examining an artificial (i.e., lumped) RC cable and an artificial RLC transmission line that extend transfinitely ("spatially beyond infinity," so to speak). We will find that nonstandard artificial diffusions and waves can "penetrate infinity" and can pass on to transfinite extensions of artificial cables and lines during unlimited hyperreal times with appreciable hyperreal values.

6 A Transfinite RLC Ladder

Let us now consider a transfinite ladder in the form of Fig. 1(a), where now every branch is a one-port consisting internally of finitely many resistors, inductors, and/or capacitors, the series branches all being the same and the shunt branches all being the same but in general different from the series branches. The only source is a voltage source $e(t)$, $t \in \mathbb{R}_+$, at the input of the ladder. Upon applying the Laplace transformation, we obtain the transformed circuit of Fig. 2(a), where Z denotes an impedance $s \mapsto Z(s)$, Y denotes an admittance $s \mapsto Y(s)$, and $E\colon s \mapsto E(s)$ is the transformed input voltage; here, s is a complex variable. Later on, we set $x = ZY$. By an *el-section* we will mean a series Z followed by a shunt Y.

To obtain a particular nonstandard version of this network, we short and open branches as shown in Fig. 1(b) to obtain a finite ladder and then restore el-sections two at a time, one for each conventional ladder, proceeding from left to right in each ladder. One stage of this restoration process is shown in Fig. 2(b), where the natural numbers $j = 0, 1, \ldots, m, m+1, \ldots, 2m$ serve as indices for the nodes that have restored incident branches.

Given $V_0 = E$, we wish to determine the resulting node voltages V_j. Perhaps the easiest way of doing this is to set $V_{2m} = 1$, compute the V_j working from right to left using Kirchhoff's and Ohm's law to get finally V_0, and then multiply all the obtained V_j by E/V_0 to get the node voltages corresponding to the input voltage E. In the following, I_j is the current in the series impedance flowing from node $j-1$ to node j. So, upon setting $V_{2m} = 1$ and $x = ZY$, we get (with $Q_0 = 1$)

$$I_{2m} = YV_{2m} = Y = YQ_0$$

$$V_{2m-1}(x) = ZY + 1 = x + 1 = P_1(x)$$

$$I_{2m-1}(x) = YV_{2m-1}(x) + I_{2m}(x) = Y(x+2) = YQ_1(x)$$

$$V_{2m-2}(x) = ZI_{2m-1}(x) + V_{2m-1}(x) = x^2 + 3x + 1 = P_2(x)$$

$$I_{2m-2}(x) = YV_{2m-2}(x) + I_{2m-1}(x) = Y(x^2 + 4x + 3) = YQ_2(x)$$

Continuing in this fashion, we obtain the following recursive equations, wherein $k = 1, 2, \ldots, 2m$.

$$P_0(x) = 1, \;\; Q_0(x) = 1$$

$$P_k(x) = xQ_{k-1}(x) + P_{k-1}(x)$$

$$Q_k(x) = P_k(x) + Q_{k-1}(x)$$

$$V_{2m} = 1, \;\; I_{2m} = Y$$

$$V_{2m-k}(x) = P_k(x)$$

$$I_{2m-k}(x) = YQ_k(x)$$

Both P_k and Q_k are polynomials in x of degree k:

$$P_k(x) \;=\; p_{k,0} + p_{k,1}x + \ldots + p_{k,k-1}x^{k-1} + p_{k,k}x^k$$

$$Q_k(x) \;=\; q_{k,0} + q_{k,1}x + \ldots + q_{k,k-1}x^{k-1} + q_{k,k}x^k$$

It follows from these recursive equations that the first and last coefficients are $p_{k,0} = 1$, $p_{k,k} = 1$, $q_{k,0} = k+1$, and $q_{k,k} = 1$. (Values for other coefficients up to $k = 10$ are listed in Table 1 of [8, page 605].) Thus, with $j = 0, 1, \ldots, 2m$ being the indices for the nodes as in Fig.2(b), we have $j = 2m - k$ (with $k = 0$ when $j = 2m$). Then, the voltage transfer function from node 0 to node j is

$$\frac{V_j(x)}{E} \;=\; \frac{V_{2m-k}(x)}{V_0(x)} \;=\; \frac{P_k(x)}{P_{2m}(x)} = \frac{1 + p_{k,1}x + \ldots + p_{k,k-1}x^{k-1} + x^k}{1 + p_{2m,1}x + \ldots + p_{2m,2m-1}x^{2m-1} + x^{2m}}, \qquad (1)$$

and the transfer admittance from the input voltage to the current in the jth series branch is

$$\frac{I_j(x)}{E} \;=\; \frac{I_{2m-k}(x)}{V_0(x)} \;=\; \frac{YQ_k(x)}{P_{2m}(x)} = \frac{Y(k+1+q_{k,1}x + \ldots + q_{k,k-1}x^{k-1} + x^k)}{1 + p_{2m,1}x + \ldots + p_{2m,2m-1}x^{2m-1} + x^{2m}}.$$

7 A Nonstandard Transfinite Artificial Cable

As our first example of a nonstandard transient regime on a transfinite network, we examine a transfinite artificial cable consisting of two conventional one-way-infinite artificial cables in cascade, one being a transfinite extension of the other, as shown in Fig. 2(a). The cables are "artificial" because they consist of lumped elements. Each series element is a resistor of value r, and each shunt element is a capacitor of value c. Thus, $x = rcs$. Let us assume that the source at the input provides a unit-step of voltage, so that $E(s) = 1/s$. Consider the standard voltage $v_j(t)$ at node j ($1 \leq j \leq 2m$) for the truncated finite ladder of Fig. 2(b). It follows from (1) and the initial-value theorem that the node voltages have, for $t \to 0+$, the asymptotic values

$$v_j(t) \sim \frac{1}{(rc)^j} \cdot \frac{t^j}{j!}, \quad j = 0, \ldots, 2m. \tag{2}$$

In fact these asymptotic expressions are also bounds on the $v_j(t)$ for all $t \in \mathbb{R}_+$ [6]:

$$|v_j(t)| \leq \frac{1}{(rc)^j} \cdot \frac{t^j}{j!}. \tag{3}$$

For each j, this bound holds for all truncations beyond node j of the transfinite ladder. Furthermore, all the poles of $V_j(s)$ are real and negative [5, page 332] except for a simple pole at the origin due to the factor $E(s) = 1/s$. Consequently, we may use (1) and the final-value theorem to write $\lim_{t\to\infty} v_j(t) = 1$ (which indeed is physically obvious).

Next, we argue that $v_j(t)$ is strictly monotonically increasing for all $t > 0$. Indeed, the voltage transfer function $V_j(s)/E(s)$ in the Laplace-transform domain can be written as a product of the driving-point admittances and impedances, shown in Fig. 2(b), measured toward the right:

$$\frac{V_j}{E} = \frac{I_1}{E} \cdot \frac{V_1}{I_1} \cdot \frac{I_2}{V_1} \cdot \frac{V_2}{I_2} \cdots \frac{I_j}{V_{j-1}} \cdot \frac{V_j}{I_j} = Y_1 Z_1 Y_2 Z_2 \cdots Y_j Z_j. \tag{4}$$

For each index i ($1 \leq i \leq j$), all the poles and zeros of Z_i are simple and alternate along the nonpositive real axis, with a pole at the origin and a zero at infinity [5, page 412]. As for the product $Y_i Z_i$, we have

$$Y_i Z_i = \frac{1}{1 + \frac{r}{Z_i}},$$

from which it follows that all the poles and zeros of $Y_i Z_i$ are simple and alternate along the negative real axis; closest to the origin is a pole, not a zero. Also, $Z_i(s)Y_i(s)$ tends to 0 as $|s| \to \infty$. Thus, the residues of the poles are positive, and the unit-impulse response of $Y_i Z_i$ is a finite sum of the form $\sum_{k=1}^{K} a_k \, e^{-\rho_k t}$, where $a_k > 0$ and $\rho_k > 0$ for all k. So, upon applying the convolution integral repeatedly according to the inverse Laplace transform of (4), we can conclude that the unit-impulse response is positive for all $t > 0$. It follows that the unit-step response for $v_j(t)$ (i.e., with $E(s) = 1/s$) is continuous and strictly monotonically increasing, as asserted.

This also true for the voltage $v_{j,\infty}(t)$ at the jth node of a one-way-infinite artificial RC cable; namely, $v_{j,\infty}(t)$ is a continuous, strictly monotonically increasing function, starting at $t = 0$ and approaching 1 as $t \to \infty$ (see the Appendix). Moreover, [1, Formula 9.6.7] implies that $v_{j,\infty}(t)$ has the asymptotic form (2).

Let us now consider the hyperreal voltage transient at a fixed node of the nonstandard transfinite artificial RC cable resulting from the restoration of el-sections two at a time, one for each of the two conventional ladders. That is, the restoration proceeds through the finite ladders of Fig. 2(b) toward the transfinite ladder of Fig. 2(a).[7] Let us number these restoration stages by $n = 0, 1, 2, \ldots$. Thus, $n = m$, and we have 2m el-sections in Fig. 2(b). When $n = 0$, we have only the source $e(t)$, which again we take to be a unit-step of voltage.

Going to the nonstandard case corresponding to the stated restoration process, let $\mathbf{v}_j(\mathbf{t})$ be the hyperreal voltage transient at the jth node in the first ladder. As $\mathbf{t}$ increases through ${}^*\!\boldsymbol{R}_+$, $\mathbf{v}_j(\mathbf{t})$ does too; indeed, for $0 < \mathbf{t}_1 = \langle t_{1,n} \rangle < \mathbf{t}_2 = \langle t_{2,n} \rangle$, we have $t_{1,n} < t_{2,n}$ a.e. and $v_{j,n}(t_{1,n}) < v_{j,n}(t_{2,n})$ a.e., so that

$$0 < \mathbf{v}_j(\mathbf{t}_1) = \langle v_{j,n}(t_{1,n}) \rangle < \langle v_{j,n}(t_{2,n}) \rangle = \mathbf{v}_j(\mathbf{t}_2). \tag{5}$$

When $\mathbf{t}$ increases through the positive infinitesimals, $\mathbf{v}_j(\mathbf{t})$ also remains infinitesimal. This follows directly from (3). Indeed, for any positive infinitesimal $\mathbf{t} = \langle t_n \rangle$, we have

$$|\mathbf{v}_j(\mathbf{t})| = \langle |v_{j,n}(t_n)| \rangle \le \frac{1}{(rc)^j j!} \langle t_n^j \rangle.$$

The right-hand side is infinitesimal.

On the other hand, as soon as $\mathbf{t}$ becomes appreciable, $\mathbf{v}_j(\mathbf{t})$ becomes appreciable, too. This follows from (2): Let $\mathbf{w} = \langle w_n \rangle$ be any positive infinitesimal. Thus, we can take $w_n \to 0+$ as $n \to \infty$. Then, for $\mathbf{v}_j(\mathbf{t}) = \langle v_{j,n}(t_n) \rangle = \mathbf{w} = \langle w_n \rangle$, we have from (2) that

[7] Let us emphasize that the nonstandard network depends in general upon the chosen restoration sequence. This issue is explored at some length in [12].

$\mathbf{t} = \langle t_n \rangle = (rc)^j j! \langle w_n^{1/j} \rangle$ is infinitesimal, too. This means that, as $\mathbf{t}$ increases through the positive infinitesimals and then becomes appreciable, $\mathbf{v}_j(\mathbf{t})$ increases through all the positive infinitesimals. By the permanence principle (see [2, page 137]), $\mathbf{v}_j(\mathbf{t})$ also achieves appreciable values. Indeed, let $\mathbf{t}$ increase throughout the internal interval $\langle I_n \rangle$, where $I_n = [0, \epsilon]$ for each n and where ϵ is any fixed positive real number. Then, $\mathbf{v}_j(\langle I_n \rangle)$ is an internal interval [2, page 148] containing all the infinitesimals, and by [2, Theorem 11.9.1], $\mathbf{v}_j(\langle I_n \rangle)$ also contains appreciable values. Since this is so for every positive real ϵ, no matter how small, our assertion follows.

Furthermore, $\mathbf{v}_j(\mathbf{t})$ is infinitesimally close to 1 (more specifically, $1 - \mathbf{v}_j(\mathbf{t})$ is a positive infinitesimal), for all sufficiently large unlimited $\mathbf{t}$. Indeed, let us choose $\mathbf{w} = \langle w_n \rangle$ as a postive infinitesimal, as before. Then, because of the continuity and strictly increasing monotonicity of $v_{j,n}$, there is, for each n, a unique T_n such that $1 - v_{j,n}(T_n) = w_n$, and $0 < 1 - v_{j,n}(t_n) \leq 1 - v_{j,n}(T_n)$ for all $t_n \geq T_n$. So, for all $\mathbf{t} = \langle t_n \rangle \geq \mathbf{T} = \langle T_n \rangle$, we have $0 < 1 - \mathbf{v}_j(\mathbf{t}) \leq 1 - \mathbf{v}_j(\mathbf{T}) = \mathbf{w}$, as asserted.

Altogether, the hyperreal transient $\mathbf{v}_j(\mathbf{t})$ behaves much like the standard transient $v_{j,\infty}(t)$.

However, for a fixed node in the second ladder, we have a different situation. Let that be the pth node therein $(0 \leq p \leq \omega)$. With respect to node numbering shown in Fig. 1(a), that node's index is $\omega + p$. As $\mathbf{t}$ increases, so too does $\mathbf{v}_{\omega+p}(\mathbf{t})$ (replace j by $\omega + p$ in (5)). As $\mathbf{t}$ increases first through the infinitesimals and then through all the appreciable hyperreals, the hyperreal voltage $\mathbf{v}_{\omega+p}(\mathbf{t})$ remains infinitesimal, even for all appreciable $\mathbf{t}$. This, too, follows from (3) because now the number of el-sections preceding the node $\omega + p$ increases indefinitely during the restoration process. Thus, for any appreciable $\mathbf{t} = \langle t_n \rangle$, there is a $T \in \boldsymbol{R}_+$ such that $0 < t_n < T$ a.e. Then,

$$0 \leq v_{\omega+p,n}(t_n) \leq \frac{1}{(rc)^{n+p}} \cdot \frac{T^{n+p}}{(n+p)!}$$

for almost all n. The right-hand side tends to 0 as $n \to \infty$, whence our assertion.

On the other hand, as $\mathbf{t} = \langle t_n \rangle$ increases through the unlimited hyperreals, $\mathbf{v}_{\omega+p}(\mathbf{t})) = \langle v_{\omega+p,n}(t_n) \rangle$ remains infinitesimal for a while but then increases through appreciable values, eventually getting infinitesimally close to 1 but remaining less than 1. For instance, for

$\mathbf{t} = \langle t_n \rangle$ and $n > 1$, we can choose

$$\frac{t_n}{rc} = [(n+p-1)!]^{1/(n+p)}. \tag{6}$$

Then, by (3) again,

$$|v_{\omega+p,n}(t_n)| \leq \frac{(n+p-1)!}{(n+p)!} = \frac{1}{n+p} \rightarrow 0, \quad n \rightarrow \infty.$$

So, $\mathbf{v}_{\omega+p}(\mathbf{t}) = \langle v_{\omega+p,n}(t_n) \rangle$ is infinitesimal. On the other hand, this choice of the t_n corresponds to an unlimited $\mathbf{t} = \langle t_n \rangle$, as can be seen from Stirling's asymptotic formula for the gamma function [1, page 257]. Indeed, the right-hand side of (6) is asymptotic to $(n+p)/e$ as $n \rightarrow \infty$.

However, for sufficiently large unlimited $\mathbf{t}$, $\mathbf{v}_{n+p}(\mathbf{t})$ becomes appreciable. To see this, let $w \in \boldsymbol{R}_+$ be such that $0 < w < 1$. Then, for each n, there is a unique t_n such that $v_{\omega+p,n}(t_n) = w$. By what we have already shown, $\mathbf{t} = \langle t_n \rangle$ must be unlimited. Also, $\mathbf{v}_{n+p}(\langle t_n \rangle) = \langle w \rangle$ is appreciable. If we replace w by w_n with $w_n \rightarrow 1-$, we get that $v_{\omega+p,n}(t_n) = w_n \rightarrow 1-$. Thus, $\mathbf{v}(\mathbf{t}) = \langle v_{\omega+p,n}(t_n) \rangle$ is infinitesimally close to 1 for all sufficiently large unlimited $\mathbf{t}$, as before.

To summarize all this heuristically, let us first note that voltage transmission along an artificial RC cable corresponds to a discrete version of diffusion. So, upon applying a unit-step of voltage at the input to the transfinite artificial cable, we have, for appreciable values of time, appreciable values of voltage artificially diffusing throughout the first ladder, no matter how small appreciable time may be. However, the second ladder (the transfinite extension) has only infinitesimal voltages, no matter how large appreciable time may be and even for some initial unlimited values of $\mathbf{t}$. It is only when hyperreal time becomes sufficiently unlimitedly large that appreciable voltage diffuses into the second ladder. Finally, at each node of both ladders, the voltages become infinitesimally close to 1 for all sufficiently large unlimited time.

Similar results can be derived for transfinite ladders that are cascades of many infinite artificial RC cables, even for higher ranks of transfiniteness. For any node, we need merely choose $\langle t_n \rangle$ as a sufficiently large unlimited hyperreal in order to get an appreciable voltage.

8 A Nonstandard Transfinite Artificial Transmission Line

The arguments needed in this section are quite similar to those of the preceding one, and therefore we will merely summarize most of them.

For this second example, we convert the artificial cable just considered into an artificial transmission line by inserting an inductor l in series with each resistor r. Thus, we now have that $x = (ls + r)cs$. With a unit-step of voltage applied at the input so that $E(s) = 1/s$, consider the voltage $v_j(t)$ at the jth node of the truncated finite ladder of Fig. 2(b). In general, $v_j(t)$ is not monotonic, although it may be if l is small enough. The initial-value theorem and (1) yield the asymptotic estimate:

$$v_j(t) \sim \frac{1}{(lc)^j} \cdot \frac{t^{2j}}{(2j)!}, \quad t \to 0+, \tag{7}$$

which also happens to be a bound on the transient for all $t \in \mathbb{R}_+$ [6]:

$$|v_j(t)| \leq \frac{1}{(lc)^j} \cdot \frac{t^{2j}}{(2j)!}. \tag{8}$$

Also, because of the presence of the series resistors, all the poles of the Laplace transform V_j of $v_j(t)$ are in the left-half s-plane [5, page 332] except for the simple pole at the origin due to $E(s)$. Thus, the final-value theorem can be applied to (1) to get $v_j(t) \to 1$ as $t \to \infty$.

Now let us restore el-sections as in the preceding section. At a fixed jth node in the first ladder $(0 < j < m)$ and with $n = m$ as the index of the restoration procedure, we have the hyperreal voltage $\mathbf{v}_j(\mathbf{t}) = \langle v_{j,n}(t_n) \rangle$ defined for hyperreal time $\mathbf{t} = \langle t_n \rangle \in {}^*\mathbb{R}_+$. By arguments similar to those given in Sec. 7, we have the following results:

It follows from (8) that $\mathbf{v}_j(\mathbf{t})$ is infinitesimal when $\mathbf{t}$ is. The permanence principle coupled with (7) shows that, as $\mathbf{t}$ increases through the positive infinitesimals and then becomes appreciable, $\mathbf{v}_j(\mathbf{t})$ also increases through the positive infinitesimals and then becomes appreciable. But now, for some larger appreciable values, $\mathbf{v}_j(\mathbf{t})$ might vary through 0 and its halo because of the possible oscillatory nature of the $v_{j,n}(t)$.

Finally, for all sufficiently large unlimited $\mathbf{t}$, $\mathbf{v}_j(\mathbf{t})$ is infinitesimally close to 1. For this last assertion, we have to adjust some inequalities to account for the fact that v_j may be oscillatory. In particular, we choose $\mathbf{w} = \langle w_n \rangle$ as a positive infinitesimal and choose T_n as the minimum time for which $|1 - v_{j,n}(t_n)| \leq w_n$ for all $t_n \geq T_n$. So, for all $\mathbf{t} = \langle t_n \rangle \geq \mathbf{T} = \langle T_n \rangle$, we have $|1 - \mathbf{v}_j(\mathbf{t})| \leq \mathbf{w}$.

Here, too, we can examine transients in the second ladder, something that could not be done heretofore when only standard analyses of transfinite networks was available. As before, let $\omega + p$ denote the index of a fixed node in the second ladder ($0 \leq p \leq \omega$). Also, as before, let us restore the transfinite ladder using two el-sections at each step n of restoration, one for each ladder. Thus, $n = m$, again as before. For a limited hyperreal $\mathbf{t} = \langle t_n \rangle$, $\mathbf{v}_{\omega+p}(\mathbf{t}) = \langle v_{\omega+p,n}(t_n) \rangle$ remains infinitesimal, as can be seen from (8) and an argument similar to that in the preceding section. However, as $\mathbf{t} = \langle t_n \rangle$ increases through the unlimited hyperreals, $\mathbf{v}_{\omega+p}(\mathbf{t})$ remains infinitesimal at first but then increases through appreciable hyperreal values and finally gets infinitesimally close to 1 for sufficiently large hyperreal values of $\mathbf{t}$. This, too, is argued as in the preceding section with some adjustment of inequalities.

To summarize, the artificial wave propagates down the first ladder with appreciable values for all appreciable values of time. However, it propagates into the second ladder with appreciable values only for sufficiently large unlimited hyperreal times.

9 Conclusions

We have shown in Sec. 5 that transient analyses can be made for transfinite RLC networks if nonstandard analysis is used. Such had not been done heretofore; standard analyses of transfinite networks have always been restricted to purely resistive ones. As examples, transient analyses for a transfinite artificial cable and for a transfinite artificial RLC transmission line are examined in Secs. 7 and 8.

Let us summarize our results more figuratively and succinctly: It is shown in this paper that it is possible to "pass beyond infinity," not only spatially along transfinite graphs—as has been shown in prior works for purely resistive networks, but also temporally in RLC networks during unlimited hyperreal time.

Appendix

The voltage transfer ratio across one el-section of a one-way-infinite artificial RC cable is is $V_j/V_{j-1} = 1/(1 + r/Z_d)$, where Z_d is the characteristic impedance. By a customary manipulation, we have

$$Z_d = -\frac{r}{2} + \sqrt{\left(\frac{r}{2}\right)^2 + \frac{r}{cs}}.$$

So, with $a = rc/2$, we get, after some manipulation involving the completion of a square,

$$\frac{V_j}{V_{j-1}} = as + 1 - \sqrt{(as+1)^2 - 1}.$$

Then, for $E(s) = 1/s$, we get

$$V_j(s) = \frac{1}{s}\left[as + 1 - \sqrt{(as+1)^2 - 1}\right]^j.$$

By Formula 90 of [7, page 354], we get the following voltage transient at node j due to a unit step of voltage at the input:

$$v_j(t) = j\int_0^t \tau^{-1}\, I_j(b\tau)\, e^{-b\tau}\, d\tau$$

where I_j is the modified Bessel function of first kind and order j and $b = 1/a = 2/rc$. $I_j(bt)$ is positive for all $t > 0$, and $I_j(bt)/t$ is asymptotic to $(b/2)^j t^{j-1}/j!$ as $t \to 0+$ for each $j \geq 1$. (See page 374 et seq. of [1].) Thus, $v_j(t)$ is continuous and strictly monotonically increasing for $t > 0$. We are justified in applying the final-value theorem. (See [7, Theorem 8.7-1] and [1, Formula 9.7.1].) This gives $\lim_{t\to\infty} v_j(t) = \lim_{s\to 0+} sV_j(s) = 1$. This establishes all the properties of $v_{j,\infty}(t)$ asserted in Sec. 7.

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Figure Captions

Fig. 1.

(a) A transfinite graph consisting of two one-way-infinite grounded ladders in cascade. The small circles denote 1-nodes. The 1-node in the middle connects the infinite extremity of the first ladder to the input of the second ladder. The upper nodes are indexed first by the natural numbers $0, 1, 2, \ldots$ and then by the transfinite ordinals $\omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2$.

(b) A finite graph obtained by shorting and opening branches in the same way for both ladders.

Fig. 2.

(a) A transfinite RLC network having the same graph as that of Fig. 1(a).

(b) The finite RLC network having the same graph as that of Fig. 1(b). Each V_j and I_j is the Laplace transform of a time-dependent node voltage and branch current. The Y_j and Z_j are driving-point admittances and impedances, respectively, for the networks to the right of the places where the Y_j and Z_j occur.

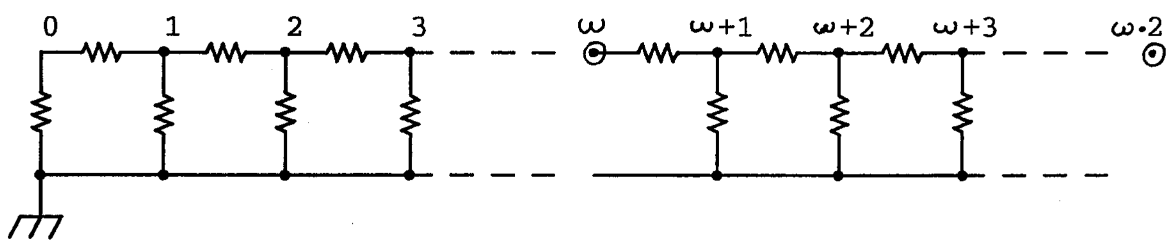


Fig. 1(a)

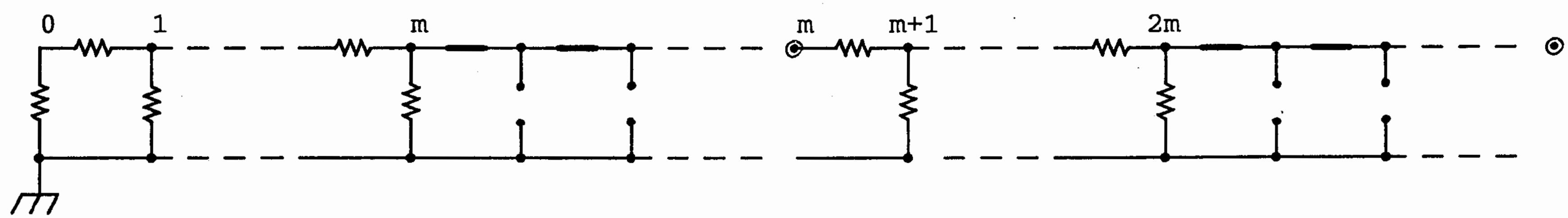


Fig. 1(b)

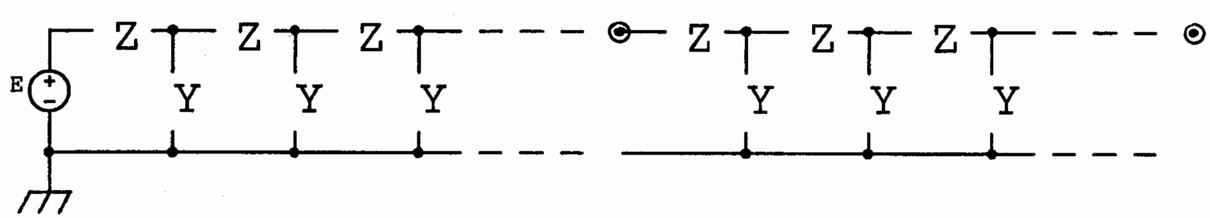
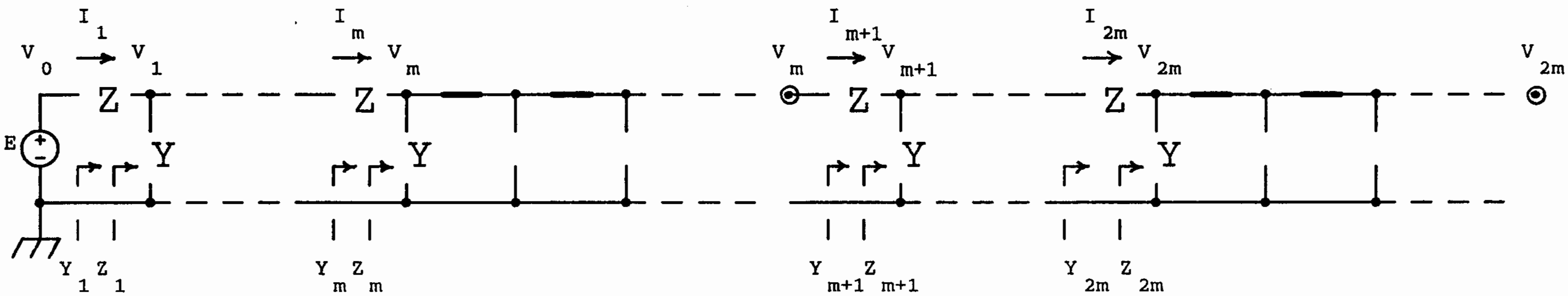


Fig. 2(a)



V 0
I 1
V 1
E
Z
Y
Y 1 Z 1
I m
V m
Y m Z m
V m
I m+1
V m+1
Y m+1 Z m+1
I 2m
V 2m
Y 2m Z 2m
V 2m

Fig. 2(b)