PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A ROOT-FINDING ALGORITHM FOR CUBICS

SAM NORTHSHIELD

Abstract. Newton's method applied to a quadratic polynomial converges rapidly to a root for almost all starting points and almost all coefficients. This can be understood in terms of an associative binary operation arising from 2×2 matrices. Here we develop an analogous theory based on 3×3 matrices which yields a two-variable generally convergent algorithm for cubics.

1. Introduction

Newton's method applied to a quadratic polynomial converges rapidly to a root (for almost all starting points and almost all choices of coefficients). Newton's method is thus said to be "generally convergent" for polynomials of degree 2. It is not generally convergent for polynomials of degree 3 however. For example, Newton's method applied to $z^3 - 2z + 2$ (or any other cubic polynomial with coefficients sufficiently close to $(1, 0, -2, 2)$) asymptotically follows a 2-cycle. McMullen [**?**] constructed a generally convergent algorithm for cubic polynomials and further showed that there is no such algorithm for higher degree polynomials. Hawkins [**?**] noted that McMullen's algorithm for $z^3 - 1$ is actually Halley's method and that all applications of McMullen's algorithm are conjugate, by Möbius transformations, to that.

The fact that Newton's method is generally convergent for quadratic polynomials and in fact converges quadratically fast (i.e., $|x_{n+1} - r| \leq C|x_n - r|^2$ for almost all starting points x_0 and for some root r) can be shown by construction of an associative binary operation arising from 2×2 matrices (see [**?**]). Here we shall review that theory and develop an analogous theory based on 3×3 matrices. Our goal is to construct a new, generally convergent algorithm for cubics (involving two variables) for which the theory is particularly simple.

In what follows, we shall assume that all matrices and coefficients have entries in $\mathbb{C}$.

2. The 2-dimensional case

Let A be a 2×2 matrix with distinct eigenvalues λ_1, λ_2, and with characteristic polynomial $p(x) := x^2 - \tau x + \delta = (x - \lambda_1)(x - \lambda_2)$ where τ is the trace of A and δ is the determinant of A. By the Cayley-Hamilton theorem, A satisfies its characteristic polynomial:

$$A^2 = \tau A - \delta I.$$

2000 *Mathematics Subject Classification.* Primary 65H04; Secondary 26C10, 30D05, 37F10.
Key words and phrases. Newton's method, iterative algorithm, generally convergent.

It follows, for any x, y where $x + y \neq \tau$,

$$(A-xI)(A-yI) = A^2-(x+y)A+xyI = (\tau-x-y)A+(xy-\delta)I \doteq A-\frac{xy-\delta}{x+y-\tau}I$$

where $M \doteq N$ means $M = kN$ for some scalar k. Letting

$$x \oplus y := \frac{xy-\delta}{x+y-\tau},$$

we have $(A - xI)(A - yI) \doteq (A - (x \oplus y)I)$ and it follows that $\oplus$ is commutative and associative. The map

$$x \mapsto x \oplus x = \frac{x^2-\delta}{2x-\tau} = x - \frac{p(x)}{p'(x)}$$

is just the application of one step of Newton's method applied to the function $p(x)$. Hence, if (x_n) is the sequence of approximants to a root of $p(x)$ by Newton's method starting at $x_0 = x$, then x_n is the 2^n-fold "sum" of x with itself under the operation $\oplus$; we write

$$x_n = x^{\oplus 2^n}.$$

Note that

$$x \oplus y - \lambda_i = \frac{xy-\delta}{x+y-\tau} - \lambda_i = \frac{xy-(x+y)\lambda_i+\tau\lambda_i-\delta}{x+y-\tau} = \frac{(x-\lambda_i)(y-\lambda_i)}{x+y-\tau}$$

and so

$$\frac{x \oplus y - \lambda_1}{x \oplus y - \lambda_2} = \frac{x-\lambda_1}{x-\lambda_2} \cdot \frac{y-\lambda_1}{y-\lambda_2}.$$

With Newton approximants (x_n) as above, if x_0 is closer to λ_1 than to λ_2 then

$$\frac{x_n-\lambda_1}{x_n-\lambda_2} = \left(\frac{x_0-\lambda_1}{x_0-\lambda_2}\right)^{2^n} \to 0$$

and thus $x_n \to \lambda_1$ quadratically.

Higher order methods arise easily. For example, Halley's method (applied to quadratic p) can be expressed in terms of $\oplus$:

$$x \mapsto x - \frac{2p(x)p'(x)}{2p'(x)^2 - p(x)p''(x)} = \frac{x^3-3\delta x+\tau\delta}{3x^2-3\tau x+\tau^2-\delta} = x \oplus x \oplus x$$

and thus converges cubically.

3. The 3-dimensional case

Let A be a 3×3 matrix with eigenvalues r, s, t. This induces an 'addition' on 2-vectors: define $(a, b) \oplus (c, d) = (u, v)$ if

$$(A^2 - aA + bI)(A^2 - cA + dI) = k(A^2 - uA + vI)$$

for some scalar k. By the Cayley-Hamilton theorem,

$$A^3 = \tau A^2 - \sigma A + \delta I \qquad (1)$$

where $\tau = r + s + t$ (the trace of A), $\delta = rst$ (the determinant of A), and $\sigma := rs + rt + st$. Hence

$$A^4 = (\tau^2 - \sigma)A^2 - (\tau\sigma - \delta)A + \tau\delta I.$$

and it follows that

$$(A^2 - aA + bI)(A^2 - cA + dI)$$

$$= A^4 - (a+c)A^3 + (b+d+ac)A^2 - (ad+bc)A + bdI$$
$$= (b+d+ac-a\tau-c\tau+\tau^2-\sigma)A^2$$
$$-(ad+bc-a\sigma-c\sigma+\tau\sigma-\delta)A + (bd-a\delta-c\delta+\tau\delta)I$$

and thus, for $R := \tau - a - c$,

$$(a,b)\oplus(c,d) := \left(\frac{ad+bc-\delta+R\sigma}{b+d+ac-\sigma+R\tau}, \frac{bd+R\delta}{b+d+ac-\sigma+R\tau}\right). \tag{2}$$

Clearly, $\oplus$ is an associative binary operation as it inherits this property from associativity of matrix multiplication. Let $(x,y)^{\oplus n}$ denote the n-fold 'sum' of (x,y) with itself by $\oplus$:

$$(x,y)^{\oplus 1} := (x,y) \text{ and, for } n \geq 1, (x,y)^{\oplus(n+1)} := (x,y)^{\oplus n} \oplus (x,y).$$

Lemma 3.1. *Let A be a matrix with distinct eigenvalues r, s, t. If x, y satisfy*

$$|r^2 - xr + y|, |s^2 - xs + y| < |t^2 - xt + y|$$

then $(x,y)^{\oplus n}$ converges to $(r+s, rs)$

Proof. Suppose $(x,y)\oplus(u,v) = (w,z)$. This means that for some k and each eigenvalue λ of A,

$$(\lambda^2 - x\lambda + y)(\lambda^2 - u\lambda + v) = k(\lambda^2 - w\lambda + z).$$

It follows that if α and β are two eigenvalues of A, then

$$\frac{\alpha^2 - x\alpha + y}{\beta^2 - x\beta + y} \cdot \frac{\alpha^2 - u\alpha + v}{\beta^2 - u\beta + v} = \frac{\alpha^2 - w\alpha + z}{\beta^2 - w\beta + z}. \tag{3}$$

Let $f(z) = z^2 - xz + y$ and define

$$(x_n, y_n) := (x,y)^{\oplus n} \text{ and } f_n(z) := z^2 - x_n z + y_n.$$

By (3),

$$\frac{f_n(r)}{f_n(t)} = \left(\frac{f(r)}{f(t)}\right)^n \text{ and } \frac{f_n(s)}{f_n(t)} = \left(\frac{f(s)}{f(t)}\right)^n \tag{4}$$

both converge to 0. Letting $\theta_n, \psi_n := (x_n \pm \sqrt{x_n^2 - 4y_n})/2$, we may use (4) to write

$$\frac{(r-\theta_n)(r-\psi_n)}{(t-\theta_n)(t-\psi_n)} \to 0 \text{ and } \frac{(s-\theta_n)(s-\psi_n)}{(t-\theta_n)(t-\psi_n)} \to 0. \tag{5}$$

If θ_n is unbounded, then there exists a subsequence $\theta_{N(n)} \to \infty$ which forces, by the first half of (5), $\psi_{N(n)} \to r$ and, by the second half of (5), $\psi_{N(n)} \to s$; a contradiction. Hence θ_n is bounded. Similarly, ψ_n is bounded and it follows that

$$r^2 - x_n r + y_n = (r-\theta_n)(r-\psi_n) \to 0 \text{ and } s^2 - x_n s + y_n = (s-\theta_n)(s-\psi_n) \to 0.$$

Therefore,

$$\begin{pmatrix} s & -1 \\ r & -1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} s^2 \\ r^2 \end{pmatrix} \text{ and so } \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} r+s \\ rs \end{pmatrix}.$$

□

We note that any "normalized" cubic polynomial $z^3 + az + b$ has distinct roots if and only if $Q := 4a^3 + 27b^2 \neq 0$ (and the roots are all real if and only if $Q < 0$).

Theorem 3.2. *If $z^3 + az + b = 0$ has distinct roots r, s, t and*

$$N(x, y) := \left(\frac{2xy - 2ax + b}{x^2 + 2y - a}, \frac{2bx + y^2}{x^2 + 2y - a} \right),$$

then for (x, y) with $|r^2 - xr + y|, |s^2 - xs + y| < |t^2 - xt + y|$, the n-th iterates of N satisfy

$$N_n(x, y) \to (r + s, rs).$$

In particular, if $(x_n, y_n) := N_n(x, y)$ then $-b/y_n \to t$ and $-x_n \to t$.

Proof. Using (1) adapted to the special case $\tau = 0, \sigma = a$, and $\delta = -b$, $N(x, y) = (x, y)^{\oplus 2}$. By the lemma, $N_n(x, y) = (x, y)^{\oplus 2^n} \to (r + s, rs)$. □

A map is termed *generally convergent* for polynomials of degree d if its iterates converge, for almost all starting points and almost all polynomials $p(x)$ of degree d, to a root of $p(x)$. Newton's method is generally convergent for degree 2 and it has been shown that there are no such algorithms for $d \geq 4$ [2, Theorem 1.1]. For $d = 3$ there exists such a method [**?**] which is Newton's method applied to $p(x)/q(x)$ for a certain quadratic q depending on p. It is also, for $p(x) := x^3 - 1$, the same as Halley's method and, for more general p, conjugate to that (see [**?**]). We note that, in general, our new methods have quadratic convergence.

Theorem 3.3. *The map N is generally convergent in the sense that for almost all starting points (x, y) and almost all a, b, for the cubic polynomials $p(x) := x^3+ax+b$, $N_n(x, y)$ converges to a point $(-t, -b/t)$ for some root t.*

Proof. For almost all a, b, $4a^3+27b^2 \neq 0$ and so $p(x)$ has distinct roots. For any pair r, s of these roots, let $m(z) := (z-r)/(z-s)$. Then $|r^2-xr+y|/|s^2-xs+y| = |m(u)| \cdot |m(v)|$ where $u, v = \frac{1}{2}(x \pm \sqrt{x^2 - 4y})$. Since $\{(u, v) : |m(u)m(v)| = 1\}$ has measure 0, so does $\{(x, y) : |r^2-xr+y| = |s^2-xs+y|\}$. Therefore, for almost all pairs (x, y), there is a choice of roots r, s, t such that $|r^2 - xr + y|, |s^2 - xs + y| < |t^2 - xt + y|$. By Theorem 1, $N_n(x, y)$ converges to $(r + s, rs)$ which equals $(-t, -b/t)$. □

A similar iterative algorithm is given by the following.

Corollary 3.4. If $z^3 + az + b = 0$ has distinct roots r, s, t then, for (x, y) with

$$|r^2 + xr - b/y|, |s^2 + xs - b/y| < |t^2 + xt - b/y|,$$

if

$$M(x, y) := \left(\frac{2bx + 2axy + by}{ay + 2b - x^2 y}, \frac{x^2y^2 - 2by - ay^2}{2x^2y - b} \right),$$

then the n-th iterates of M satisfy

$$M_n(x, y) \to (t, t).$$

Proof. By conjugation by $(x, y) \mapsto (-x, -b/y)$ (which is idempotent) and the fact that $r + s = -t$ and $rs = -b/t$. □

When $x = y$,

$$M(x, x) = \left(\frac{3bx + 2ax^2}{2b + ax - x^3}, \frac{x^4 - ax^2 - 2bx}{2x^3 - b} \right)$$

and we note that the first coordinate is the map for Newton's method for $p(x)/x^2$ and the second coordinate is the map for Newton's method for $p(x)/x$. If, in addition, $a = 0$, then the second coordinate is $T_p(x) := (x^4 - 2bx)/(2x^3 - b)$ where

$T_p(x)$ is McMullen's superconvergent algorithm for $p(x) = x^3 + b$ (see [**?**]), which by [**?**], is known to converge cubically..

Example 3.5. To illustrate the speed of convergence of the algorithm of Corollary 3.4 when $a = 0$, we approximate $\sqrt[3]{2}(= 1.25992105...)$ by iterating M starting at (2,3):

(2,3)
(.875,1.263157895)
(1.213245033, 1.309248555)
(1.260547998, 1.259900272)
(1.259920953, 1.259921154)
(1.259921050, 1.259921050)

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Department of Mathematics, SUNY, Plattsburgh, NY 12901
Current address: Department of Mathematics, SUNY, Plattsburgh, NY 12901
E-mail address: northssw@plattsburgh.edu