Complex Descartes Circle Theorem

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Abstract

We present a short proof of Descartes Circle Theorem on the "curvature-centers" of four mutually tangent circles. Key to the proof is associating an octahedral configuration of spheres to four mutually tangent circles. We also prove an analogue for spheres.

It can be traced back to at least Descartes that four mutually tangent circles have curvatures (reciprocals of radii) satisfying the relation

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2). \tag{1}$$

The Monthly has published several papers concerning this fascinating topic: [1, 3, 4, 5, 6]. It was only in 2001 [3] that is was noticed, and proved, that the "curvature-centers" (curvature times center where the center is considered a complex number) satisfy the same relation. We present a short proof of this result (Theorem 1) and an analogous version for spheres (Corollary 1).

For the purposes of this paper, a *sphere* will alway be contained in the half-space $\mathbb{C} \times [0, \infty)$ and be tangent to the complex plane. Let $S(z, r)$ denote the sphere with radius r tangent to $\mathbb{C}$ at z. It is obvious that $S(z, r)$ and $S(w, s)$ are tangent to each other if and only if

$$|z - w|^2 = 4rs.$$

It is also immediate that given any three points $z_1, z_2, z_3 \in \mathbb{C}$, there are unique numbers r_1, r_2, r_3 such that the spheres $S(z_i, r_i)$ are mutually tangent. In particular, if $\{i, j, k\} = \{1, 2, 3\}$ then

$$r_i = \frac{|z_i - z_j| \cdot |z_i - z_k|}{2|z_j - z_k|}. \tag{2}$$

We say that two circles are orthogonal if they intersect at right angles; see Figure 1.

Lemma 1. *Let C_1, C_2 be two orthogonal circles which intersect at w_1, w_2 and that have curvatures c_1, c_2 and centers z_1, z_2 respectively. Let k_1, k_2 be the curvatures of any two tangent spheres tangent to $\mathbb{C}$ at w_1, w_2 respectively. Then*

(*a*) $$k_1 k_2 = \frac{4}{|w_1 - w_2|^2} = c_1^2 + c_2^2$$

and

$$(b) \qquad k_1k_2w_1w_2 = \frac{4w_1w_2}{|w_1-w_2|^2} = c_1^2z_1^2 + c_2^2z_2^2.$$

Proof. Let $S(w, r)$ be any sphere tangent to both $S(w_1, r_1)$ and $S(w_2, r_2)$ where $r_i = 1/k_i$ for $i = 1, 2$. Then, by (2),

$$k_1k_2 = \frac{2|w_2-w|}{|w_2-w_1||w-w_1|}\frac{2|w_1-w|}{|w_2-w_1||w-w_2|} = \frac{4}{|w_1-w_2|^2}.$$

The quadrilateral $[z_1, w_1, z_2, w_2]$ in Figure 1 has area represented both as r_1r_2 and as $|z_1 - z_2||w_1 - w_2|/2$. Hence,

$$c_1^2 + c_2^2 = \frac{r_1^2+r_2^2}{(r_1r_2)^2} = \frac{|z_1-z_2|^2}{|z_1-z_2|^2|w_1-w_2|^2/4} = \frac{4}{|w_1-w_2|^2}$$

and so (a) is shown.

Without loss of generality, $z_2 - z_1, i(w_2 - w_1) \in \mathbb{R}$. Let z and a be as labelled in Figure 1. In particular, let a denote half the distance between w_1 and w_2. Then, by part (a),

$$a^2 = |w_1-w_2|^2/4 = 1/(k_1k_2) = 1/(c_1^2+c_2^2). \tag{3}$$

Referring again to Figure 1,

$$z_1 = z - \sqrt{r_1^2-a^2}, z_2 = z + \sqrt{r_2^2-a^2}, w_1 = z + ia, w_2 = z - ia, \tag{4}$$

and so, using $c_1 = 1/r_1$ and equation (3),

$$c_1^2\sqrt{r_1^2-a^2} = c_1\sqrt{1-a^2c_1^2} = \frac{c_1c_2}{\sqrt{k_1k_2}} = c_2^2\sqrt{r_2^2-a^2}.$$

Hence, using (4),

$$c_1^2z_1^2 = c_1^2z^2 - 2zc_1^2\sqrt{r_1^2-a^2} + 1 - c_1^2a^2,$$
$$c_2^2z_2^2 = c_2^2z^2 + 2zc_2^2\sqrt{r_2^2-a^2} + 1 - c_2^2a^2$$

and thus

$$c_1^2z_1^2 + c_2^2z_2^2 = (c_1^2+c_2^2)z^2 + 1 = k_1k_2(z^2+a^2) = k_1k_2w_1w_2$$

which shows part (b). □

Lemma 2. *Given three mutually tangent circles C_1, C_2, C_3 with curvatures c_1, c_2, c_3 and respective centers z_1, z_2, z_3, let c and z be the curvature and center respectively of the circle orthogonal to each of the three given circles. For $i \neq j \in \{1, 2, 3\}$, let $\{z_{ij}\} = C_i \cap C_j$, let S_{ij} denote the unique sphere tangent*

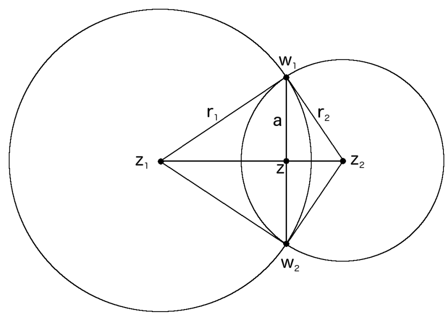

Figure 1: Two Orthogonal Circles

at z_{ij} *such that all three spheres* S_{12}, S_{13}, S_{23} *are mutually tangent, and let* k_{ij} *denote curvature of* S_{ij}*. Then*

(a) $$k_{ij} = c_i + c_j, \text{ and}$$

(b) $$c^2 z^2 = c_1 z_1 c_2 z_2 + c_1 z_1 c_3 z_3 + c_2 z_2 c_3 z_3.$$

Proof. By Coxeter [2, p. 15], $c = \sqrt{c_1 c_2 + c_1 c_3 + c_2 c_3}$. By Lemma 1,

$$c^2 + c_1^2 = k_{12} k_{13}, c^2 + c_2^2 = k_{12} k_{23}, c^2 + c_3^2 = k_{13} k_{23},$$

and so $k_{ij} = c_i + c_j$ and part (a) is shown.

Given two tangent circles with centers z, w and respective radii r, s, it is easy to see that the tangent point is $(sz + rw)/(r + s)$. Hence,

$$z_{ij} = \frac{c_i z_i + c_j z_j}{c_i + c_j} \tag{5}$$

and so, by part (a),

$$k_{ij} z_{ij} = c_i z_i + c_j z_j.$$

By Lemma 1b,

$$c^2 z^2 + c_3^2 z_3^2 = k_{13} k_{23} z_{13} z_{23} = (c_1 z_1 + c_3 z_3)(c_2 z_2 + c_3 z_3)$$

and therefore

$$c^2 z^2 = c_1 z_1 c_2 z_2 + c_1 z_1 c_3 z_3 + c_2 z_2 c_3 z_3$$

which shows part (b). □

Given four mutually tangent circles, there are six points of tangency between them (see Figure 2). At each such point z, assign a sphere tangent to the plane at z with curvature equal to the sum of the curvatures of the two circles that meet at z. By Lemma 2(a), the six spheres have disjoint interiors and any two are tangent to each other if and only if their points of tangency to the plane are on the same circle. We will refer to this collection of spheres as the *octahedral arrangement of spheres* associated with the four circles since the adjacency graph of these six circles forms an octahedron.

A key idea for the proof of the main theorem is that the respective octahedral arrangements of spheres associated to $\{C_i\}$ and to $\{C_i'\}$ are the same. We now show the Complex Descartes Circle Theorem.

Theorem 1. *Given four mutually tangent circles with curvatures c_i and respective centers z_i,*

$$\left(\sum c_i z_i\right)^2 = 2 \sum c_i^2 z_i^2.$$

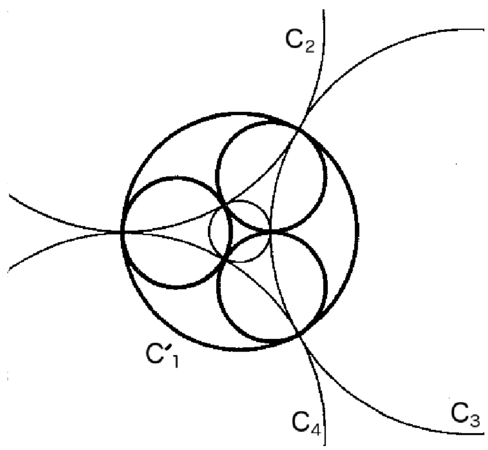

Figure 2: Four circles and their duals; three circles and their dual are labelled.

Proof. Consider a configuration of four mutually tangent circles C_1, C_2, C_3, C_4. We define C_1' to be the circle containing the three tangency points of C_2, C_3, C_4, we define C_2', C_3', C_4' similarly, and we let c_i' denote the curvature of C_i'. These form what is called the *dual configuration*, see Figure 2, where C_1', C_2, C_3 and C_4 are labelled.

For $i \neq j \in \{1, 2, 3, 4\}$, let S_{ij} be the sphere tangent to the plane at $C_i \cap C_j$ with curvature $k_{ij} := c_i + c_j$. By Lemma 2(a), the spheres $\{S_{ij} : 1 \leq i < j \leq 4\}$ form the octahedral arrangement of spheres associated with the circles $C_1, ..., C_4$. Similarly, let S_{ij}' be the sphere tangent to the plane at $C_i' \cap C_j'$ with curvature $k_{ij}' := c_i' + c_j'$.

If $\{i, j, m, n\} = \{1, 2, 3, 4\}$, then $S_{ij} = S_{mn}'$ and thus $k_{ij} = k_{mn}'$. By (5), $z_{ij}' = z_{mn}$ and thus

$$c_i' z_i' + c_j' z_j' = (c_i' + c_j') z_{ij}' = (c_m + c_n) z_{mn} = c_m z_m + c_n z_m.$$

For convenience, let $w_i = c_i z_i$, $w'_i = c'_i z'_i$, $K_{ij} = k_{ij} z_{ij}$, and $K'_{ij} = k'_{ij} z'_{ij}$. By Lemma 2(b), $w'_4 = \sigma\sqrt{w_1 w_2 + w_1 w_3 + w_2 w_3}$ (where σ is either 1 or -1) since C'_4 is the incircle of the triangle connecting the centers of C_1, C_2, and C_3. Hence

$$\begin{aligned} 2(w_1 + w_2 + w_3 - w_4) &= K_{12} + K_{34} + K_{12} + K_{13} + K_{23} - K_{14} - K_{24} - K_{34} \\ &= K'_{34} + K'_{12} + K'_{34} + K'_{24} + K'_{14} - K'_{23} - K'_{13} - K'_{12} \\ &= 4w'_4 = 4\sigma\sqrt{w_1 w_2 + w_1 w_3 + w_2 w_3}. \end{aligned}$$

It follows that

$$w_4 = w_1 + w_2 + w_3 + 2\sigma\sqrt{w_1 w_2 + w_1 w_3 + w_2 w_3}$$

and the result follows from the fact that equation (1) is equivalent to

$$d = a + b + c \pm 2\sqrt{ab + ac + bc}.$$

□

The Complex Descartes Circle Theorem is a true generalization of the Descartes Circle Theorem since replacing z_i by $(z_i + z)/z$ in the formula and taking the limit as z goes to infinity gives the old version.

The spheres in an octahedral arrangement associated with four mutually tangent circles obeys a similar formula (the proof of which follows immediately from Theorem 1).

Corollary 1. *Given four mutually tangent circles C_1, C_2, C_3, C_4, and $i \neq j \in \{1, 2, 3, 4\}$, let $\{z_{ij}\} = C_i \cap C_j$, S_{ij} be the sphere tangent to plane at z_{ij} with curvature $k_{ij} := c_i + c_j$. Then,*

(*a*) $$2\left(\sum_{i<j} k_{ij}\right)^2 = 9\sum_{i<j} k_{ij}^2,$$

(*b*) $$2\left(\sum_{i<j} k_{ij} z_{ij}\right)^2 = 9\sum_{i<j} k_{ij}^2 z_{ij}^2.$$

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