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Symmetry Analysis of the Modified Emden Equation

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Objective

- Emden equation- what is it?
- What is symmetry?
- What is a symmetry in the context of differential equations?
- How can we find symmetry?
- What are the symmetries of Emden?
- How can we use symmetry to our advantage?

Bio and Importance

- Johnathan Lane(1819-1880) and Robert Emden (1862-1940) used the Lane-Emden equation to describe phenomena in outer space.

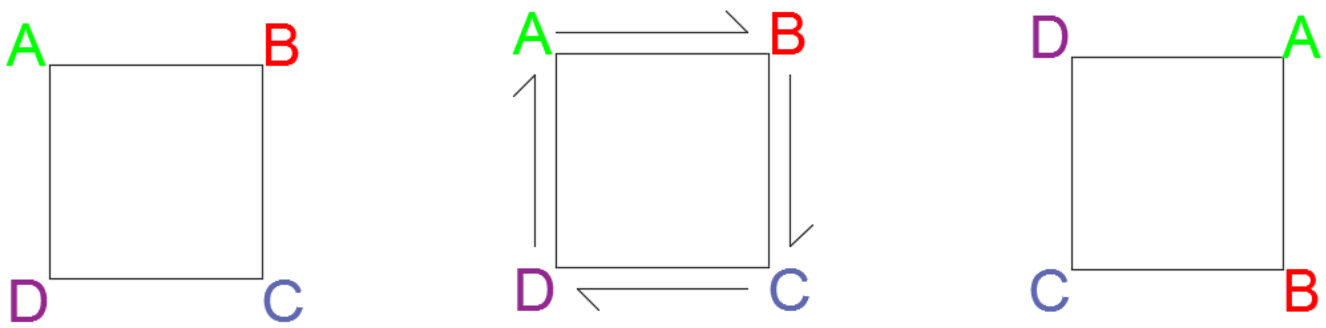
- Relates to nonlinear dynamics(force free Duffing oscillator), predator-prey models(Lotka-Volterra), and elsewhere.

- Modified Emden : $y'' + 3kyy' + y^3 = 0$

What is Symmetry?

- Rotations and Flips:

R90, R180, R270, R360, V, H, D1,D2



- Symmetries form a group
- Groups have the following properties:
- Closure, Associativity, Identity, and Invertibillity

Invariance of Differential Equations

- A symmetry will lead to a transform that leaves the equation invariant and transforms the solution to another solution.
- $\frac{dy}{dx} = xy^3, (x, y) \Rightarrow (\bar{x}, \bar{y}), \bar{x} = xe^{\varepsilon}, \bar{y} = ye^{-\varepsilon}$
- Will show that $\frac{d\bar{y}}{d\bar{x}} = \bar{x}\bar{y}^3$ holds when original equation holds

LHS: $\frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}}{dx} * \frac{dx}{d\bar{x}} = \frac{d\bar{y}}{dx} * \frac{1}{\frac{d\bar{x}}{dx}} = y'e^{-\varepsilon} * \frac{1}{e^{\varepsilon}} = y'e^{-2\varepsilon} = e^{-2\varepsilon}\frac{dy}{dx}$

RHS: $\bar{x}\bar{y}^3 \Rightarrow xe^{\varepsilon} * (ye^{-\varepsilon})^3 \Rightarrow xy^3e^{-2\varepsilon}$

- Result:
- $e^{-2\varepsilon}\frac{dy}{dx} = xy^3e^{-2\varepsilon} \Rightarrow \frac{dy}{dx} = xy^3$

Symmetry Transformation

- *General form of transformation:*
- $(x, y) \Rightarrow (\bar{x}, \bar{y}\ ),\ \bar{x} = x + \xi(x, y)\varepsilon + O(\varepsilon^2),\ \overline{y} = \mathrm{y} + \eta(x, y)\varepsilon + O(\varepsilon^2)$

Where

$$\frac{d\bar{x}}{d\varepsilon} = \xi|_{\varepsilon=0} = \xi(x, y)$$

$$\frac{d\bar{y}}{d\varepsilon} = \eta|_{\varepsilon=0} = \eta(x, y)$$

This shows us how the infinitesimals (represented by Gamma) and symmetries [how (x,y) → $(\bar{x}, \bar{y}\ )$] are related.

- Gamma $\Gamma = \xi \mathrm{dx} + \eta dy$

Lie's Invariance Condition- An Explanation without derivation

- Lie's Invariance condition is also known as the linearized symmetry condition.
- It is a formula allowing us to compute symmetries [actually ξ(x,y) and η(x,y)]of the differential equation.
- Comes from plugging the general form of the infinitesimal group and requiring invariance of the DE
- Condition for first order differential equations:

  $y'(x) = f(x, y)$
- $\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 = \xi\omega_x + \eta\omega_y$

Lie's Invariance Condition, Second Order

- *Consider the Emden equation to be rewritten as:*

  $y'' = -3kyy' - y^3$

- $\eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} + 2\xi_{xy})\,(y')^2 - (\xi_{yy})\,(y')^3 - 3(\xi_y)\, yy' + (\eta_y - 2\xi_x - 3\xi_y y)\,\omega = \xi\ \ \omega_x + \eta\,\omega_y + [\eta_x + (\eta_y - \xi_x)y' - \xi_y (y')^2]\,\omega_{y'}$

Where $y'' = \omega(x, y, y')$

$$\omega = -3kyy' - y^3$$

$$\omega_y = -3ky' - 3y^2$$

$$\omega_{y'} = -3ky$$

$$\omega_x = 0$$

Solving for Emden Symmetries

- η_{xx}+($2\eta_{xy}$-ξ_{xx}) y' +(η_{yy}+$2\xi_{xy}$) $(y')^2$ -(ξ_{yy}) $(y')^3$-3(ξ_y) $yy' + (\eta_y$-$2\xi_x$-$3\xi_y y)(-3kyy' - y^3) = \eta\,(-3ky' - 3y^2) + [\eta_x + (\eta_y - \xi_x)y' - \xi_x (y')^2](-3ky)$

Now we collect coefficients from each y'

c : $\eta_{xx} - (\eta_y - 2\xi_x)y^3 + 3\eta\, y^2 + 3k\eta_x y = 0$

y' : $2\eta_{xy} - \xi_{xx} - 3k\,\xi_x\, y - 3\xi_y\, y^3 + 3k\,\eta = 0$

$(y')^2$: $\eta_{yy} + 2\xi_{xy} + 6k\xi_y\, y = 0$

$(y')^3$: -(ξ_{yy}) $= 0$

We will systematically solve for ξ and η by collecting each coefficient of y', and then y within the y' equation.

Xi Solving

- Since $(\xi_{yy}) = 0$, ξ is linear in y meaning it must be in the form: $\xi = a(x)y + b(x)$

- Then:
- $\xi_x = a'y + b'$
- $\xi_y = a$
- $\xi_{xy} = a'$
- $\xi_{xx} = a''y + b''$

Eta Solving

- Now we plug in what we know:
- $(y')^2$: $\eta_{yy} - 2\xi_{\ xy} + 6k\xi_{\ y}\, y = 0$
- $\eta_{yy} = 2\xi_{\ xy} - 6k\xi_{\ y}\, y$
- $\eta_{yy} = 2a' \ \ -6kay$
- $\eta_y = 2a'\mathrm{y} \ \ -3kay^2 \ + c$
- $\eta \ = a'y^2 \ -kay^3 \ + c + d$
- $\eta_x = a''y^2 \ -ka'y^3 \ + c' + d'$
- $\eta_{xy} = 2a''\mathrm{y} \ \ -3ka'y^2 \ + c'$
- $\eta_{xx} = a'''y^2 \ -ka''y^3 \ + c'' + d''$

REMEMBER:
a(x)
b(x)
c(x)
d(x)

Plug Everything In!

y': $2\eta_{xy} - \xi_{xx} - 3\alpha\, \xi_x\, y \; - 3\xi_y\, y^3 + 3\alpha\, \eta = 0$

$c \quad : \eta_{xx} - (\eta_y - 2\xi_x) y^3 + 3\eta\, y^2 + 3k\eta_x y \quad = 0$

We plug everything in and collect the remaining coefficients:

$y^3 y'$: $3a(1 - k^2) = 0$

yy' : $(3a'' + 3kb + 3kc) = 0$

y': $(-\mathrm{b}'' + \mathrm{c}' + 3\mathrm{kd}) = 0$

y^3: $2ka'' + 2c + 2b' = 0$

y^2: $a''' - 3k\mathrm{c} + 3\mathrm{d} = 0$

y : $c'' + 3k\mathrm{d}' = 0$

C : $d'' = 0$

REMEMBER:
a(x)
b(x)
c(x)
d(x)

$if\ k \neq \pm 1\ then$ $3a(1 - k^2) = 0$ implies that a=0. This alters the symmetry because we can imply some terms are equal to 0, thus reducing symmetry.

Reminder Emden eq: $y'' + 3kyy' + y^3 = 0$

Now We Build

- Based on $d'' = 0$, we can assume a solution:
- $d = d_1 x + d_2$
- Then
- $y : c'' = 3kd' \Rightarrow c = \frac{3k}{2} d_1 \, x^2 + c_1 x + c_2$
- And so on

Determining Equations

- $if\ k = \pm 1$
- $d = d_1 x + d_2$
- $c = -\frac{3k}{2} d_1\ x^2 + c_1 x + c_2$
- $b = -\frac{k}{2} d_1\ x^3 + \frac{1}{2}(2c_1 + 3d_2)x^2 + b_1 x + b_2$
- $a = -\frac{k}{4} d_1\ x^4 - \frac{1}{2}(c_1 + d_2)x^3 - (b_1 + c_2)x^2 + a_1 x + a_2$
- *These equations are logically consistant throughout the system,*

Since a isn't 0. Notice that each iteration of constants adds two additional symmetries, meaning a total of 8 symmetries

k= ± 1

- $\eta = a'y^2 - kay^3 + c + d$
- $\xi = ay + b$
- if k= ± 1 then

$$\eta = -(c_1x^4 + c_2x^3 + c_3x^2 + c_4x + c_5)y^3 + (4c_1x^3 + 3c_2x^2 + 2c_3x + c_4)y^2 - (6c_1x^2 + (6c_2 + 2c_6)x + c_7 + 2c_3)\,y + 4c_4 + c_6 + c_1x$$

$$\xi = (c_1x^4 + c_2x^3 + c_3x^2 + c_4x + c_5)y - 2c_1x^3 + c_6x^2 + c_7x + c_8$$

8 Infinitesimals for k = ± 1

Γ_1: dx

Γ_2: $(y)dx + (-y^3)dy$

Γ_3: $(x)dx + (-y)dy$

Γ_4: $(x^2)dx + (-2xy + 2)dy$

Γ_5: $(xy)dx + (-xy^3 + y^2)dy$

Γ_6: $(x^2y)dx + (-x^2y^3 + 2xy^2 - 2y)dy$

Γ_7: $(x^3y)dx + (-x^3y^3 + 3x^2y^2 - 6xy + 4)dy$

Γ_8: $(x^4y - 2x^3)dx + (-x^4y^3 + 4x^3y^2 - 6x^2y + 4x)dy$

$k \neq \pm 1$

- *if* $k \neq \pm 1$ *and* $a = 0$*, then most terms drop:*

$$\eta = -c_2 y$$
$$\xi = c_1 + c_2 x$$

Γ_1: dx

Γ_2: $\mathrm{xdx} - ydy$

Γ_3: $(\mathrm{x} + 1)\mathrm{dx} - ydy$

Change of Variables

- First we change the variables using the formulas for the symmetry Γ_2: $\mathrm{xdx} - ydy$:

$$\xi \frac{dr}{dx} + \eta \frac{dr}{dx} = 0 \; , \frac{dx}{\xi} = \frac{dy}{\eta}$$

$$\xi \frac{ds}{dx} + \eta \frac{ds}{dx} = 1, \; \frac{dx}{\xi} = \frac{dy}{\eta} = ds$$

$$\int \frac{dx}{x} = \int \frac{-dy}{y} \Rightarrow \ln(x) = -\ln(y) + c(x, y) \Rightarrow x = \frac{r}{y}$$

$$\int \frac{-dy}{y} = \int ds \Rightarrow s = -\ln(y) \Rightarrow y = e^{-s(r)}$$

$$x = re^{s(r)}, y = e^{-s(r)}$$

New Equation

- $\frac{dy}{dx} = \mathrm{f(x, y, y', y'')}$ reduces to $\frac{ds}{dr} = \mathrm{f(r, s', s'')}$

$$y'' + 3kyy' + y^3 = 0 \; to$$

$$\frac{2(s')^2 - s''}{e^{3s}(rs'+1)^2} + \frac{rs's''+s'}{e^{3s}(rs'+1)^3} + \frac{ks'}{e^{3s}(rs'+1)} + e^{-3s} = 0$$

$$\frac{2(s')^2 - s''}{(rs'+1)^2} + \frac{rs's''+s'}{(rs'+1)^3} + \frac{ks'}{(rs'+1)} + 1 = 0$$

Reduction of Order

$$(r, s'(r)) \Rightarrow (t, v(t)\ )$$

$$v' = 3tv^3 + 2v^2 + (1 - 3k(tv\ + 1)^2)v + (tv\ + 1)^3$$

Which is a first order ODE.

Why is this interesting?

- We can transform an equation into a different equation that might be easier to solve analytically or numerically.

Thank You!

- References:

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END

- Questions?