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Area Awareness: A Preadolescent Perspective
A Study on the Geometric Thought Process in the Middle School Mathematics Classroom

By

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I. Introduction

This research examines how students solve area comparison problems in a middle school mathematics classroom. Determining the area of a polygon can be as easy as plugging numbers into a formula and using a calculator to find the results. What happens when the formula is not given or there is no particular formula for the geometric figure that is shown? Would students then be unable to answer the question, or would students dig deep into their knowledge of geometry and use the basic and defining laws of area to find the answer to the problem? The previous questions have sparked interest in the geometric thought process of students.

While in a graduate level class at SUNY Fredonia that was exploring technology in mathematics, a group of students were posed a geometry question regarding the areas of two different shapes. This group of students used a SmartBoard and geometry software to show that one of the shapes was numerically and graphically larger than the other shape. The group determined that it was easier to see which figure was larger by color-coding the interiors of the figures and placing one on top of the other. This allowed them to see which shape was visible on the outer edges. Their solution seemed simpler than another option for solving the problem: calculating distances and using a formula to determine the area for both figures and comparing the two numeric answers.

The professor in this mathematics education course posed the following question to the class after the group completed the problem: “At what grade level do you think students stop using pictures and graphs and start relying only on formulas and equations?” The class agreed that this student development occurs at some point during the middle school grade levels. The graduate student presentation and brief discussion in class was the basis for the current research.
The researcher thought about this presentation and brief discussion and developed an interest in the topic. The focus of the current research is not necessarily in which grade students make the change from using visual aids to using formulas alone, but how middle school students solve area problems in general. Conventionally, it seems that most students (and most people in general), when given a basic rectangle or square, can find the area quite simply. However, if someone is given a parallelogram, a trapezoid, or a shape that has many angles and sides, the result may be confusion without a formula to guide this person to the right answer.

This is what really attracted the researcher to this topic. The goal was to find insight into how good students are at comparing the size (area) of different shapes without any formulas given to them. The research not only examined whether or not the students were correct, but also at what problem-solving techniques the students used and how accurate their strategies were. More specifically,

*It is hypothesized that middle school students will rely on their memory and on area formulas to compare the sizes of two different geometric figures with measurements given in the diagrams, rather than an informal approach of "manipulating" the figures so that they can be visually or more easily numerically compared. Furthermore the informal approach will be more accurate than the approach that uses formulas.*

The graduate class decided that students stop using pictures and graphs and start relying heavily on formulas during the pre-adolescent stage of development; therefore a middle school mathematics classroom was chosen to test this hypothesis.

Informal manipulation, if done properly, can be formally proven and is the basis for area formulas; therefore the informal approach was hypothesized to be more accurate. There is
speculation that students who use informal problem-solving skills would not need formulas if they were able to show or prove how to find the area of a certain shape, and would then be able to present the correct answer more often than students who rely on formulas.

Many experts in the field of mathematics have experimented and researched the different aspects of geometry, while other mathematicians have sought to better understand the problem solving techniques of students. Previous research lends support to the current hypothesis.

II. Literature Review

Pictures, models, and a hands-on approach seem to work best for students at the elementary level. However, as students get older it is likely they will trade in the conception of physical manipulatives for formulas and equations. It is believed that this “trading” in of thought processes is harmful to the actual understanding and learning of students. Battista (1982) states that an undesirable consequence of learning area through formulas without learning the meaning and concepts behind it creates a false knowledge of the area concept. When dealing with area, the problem solving abilities of a student can be affected by how the student first learned about area.

Approaches to Teaching Area of Polygons

Every teacher has a different way of helping students understand the material in a course. The way information is presented to a student affects how that student learns and forms ideas. Many different ways of teaching and learning areas of polygons have been shown in many different textbooks and articles (Battista, 1982; Chambers, 1996; Collins, Dritsas, Frey, Howard, McClain, … & Wilson, 2001; Elia, Gagatsis & Kyriakides, 2003; Hoosain, 2010; Kestner & Others, 1997; Larson, Boswell, Kanold & Stiff, 2007a, 2007b).
Textbooks usually adequately provide formulas, but describing and defining area is a different story. Glencoe published a text (Collins, Dritsas, Frey, Howard, McClain, … & Wilson, 2001) for 6th graders that shows how area is actually defined. It starts by defining area with irregular shapes. Because the text uses irregular shapes, rather than squares and rectangular shapes for which students have to find the number of squares that fit inside, it offers a great way for students to grasp the area concept.

*Mathematics: Applications and Connections* (Collins, Dritsas, Frey, Howard, McClain, … & Wilson, 2001) introduces students to the concept of area by having them trace their own hand and approximate the area of their hand by counting the squares inside the tracing (see figure 1).

![Figure 1. Mathematics: Applications and Connections introduces the area concept by drawing the general shape of a hand and estimating how large it is.](image-url)
The approach that is taken in Larson, Boswell, Kanold and Stiff's (2007a) *Math: Course 2* is similar, in which the area concept is introduced, or possibly re-introduced, by finding the area of a parallelogram placed on a grid made of squares (see Figure 2). The parallelogram is a good starting point but it can easily be converted into a rectangle in which squares fit neatly. Once in rectangular form, students can easily perceive area as the multiplication of 2 numbers, length and width, instead of the core definition of how many unit-squares the figure covers, without gaps and overlaps.

*Figure 2. Math Course 2* shows how to find area of a parallelogram on a grid.

The way students learn how to calculate area affects how they solve area problems. If the student does not obtain the real meaning of area, they will be at a loss when tested on the areas of
irregular shapes, a hand as an example. Understanding a topic and learning to make calculations is essential for the student’s ability to problem solve.

**Problem Solving**

Many students solve problems the way they were shown, whether it is because they can only think of one way or because they want to do it the way the teacher does. However there are some students who think differently, which brings up the debate on how students solve problems (Battista, 1989; Contreras & Martinez-Cruz, 2009; Dunlop, 1977; Greeno, 1979; Greeno, Magone & Chaiklin, 1979; McLeay, 2006).

In the context of finding the area of polygons, there are two main ways to determine if shapes are the same size. There is the formal way, where students use equations and formulas, or the informal way where the students use reasoning and pictures to solve for area. The equations and formulas are actually derived from the informal proof methods. With this in mind, it is generally believed that if one is able to understand the informal methods, then one will have a better understanding of the concepts that are being assessed. Experts have examined the relation of a student’s cognitive development and the problem solving technique of choice.

In a study, Dunlop (1977) found that low-ability children operate at the concrete level of Piaget’s theory of cognitive development, where students solve problems using concrete objects. This stage would suggest that a student categorized as being at the concrete level would most likely try to memorize a formula and apply it to find the area of a non-regular polygon. However, some students are able to use their knowledge of other shapes and abstract ideas, most likely using a combination of diagrams and logical reasoning to draw conclusions about the area of a non-regular polygon. Students with this problem-solving ability are considered to be in the
formal operational stage of Piaget’s theory of cognitive development because they have developed the capacity to solve problems that are abstract in nature. Dunlop (1977) concluded that students who work through problems with formal operations have a high problem solving ability.

A way to determine how students solve area problems, and if their method is effective in determining a correct answer, is by testing them. Some simple tests may only show if students can follow directions, but a well-structured question has the ability to test students on a deeper level of understanding the area concept.

**Testing the Area Concept**

Although it is difficult to test for deep understanding as compared to just asking for the correct answer, it can be done. On state tests, and even classroom tests, students have been asked to express their understanding of area through the use of pictures and explanations, (NYSED, 2005a; NYSED, 2005b; NYSED, 2005c; NYSED, 2005d; NYSED, 2005e; NYSED 2005f).

The 7th grade New York State mathematics sample test from 2005 shows both testing for deeper understanding of area and testing for computational fluency. In problem 24 (see figure 3) there is a glimpse of what testing for a deeper understanding of area looks like.
What is the area of the rectangle drawn on the coordinate plane shown below?

F 21 square units
G 24 square units
H 28 square units
J 32 square units

Figure 3. Problem 24 on the 2005 NYS 7th grade mathematics test provides no area formulas.

In the figure, there is no formula to calculate area, which would suggest that students have to know the formula and figure out the measurements of the rectangle, or just count the squares that make up the inside of the figure. If the student counts the squares on the inside of the figure, they display some basic knowledge of the definition of area – at least in the sense of a figure that easily fits small squares within it.

However, later in the exam a question is asked about the surface area of a rectangular box. The dimensions of the rectangular box are given, along with the formula for surface area that is listed at the beginning of the exam. Even though this is a 3-dimensional object, the question asks about the addition of the areas of the 2-dimensional figures which comprise the sides/faces of the rectangular box. Students do not even have to know what surface area means to answer this question; they just have to replace symbols in a formula with the dimensions of the box.
Even as early as 6th grade, area has been tested on the state level. These state tests can be linked to a conversion of knowing how to find area to simply using a formula. As seen from problem 17 of the 6th grade sample test from 2005, students are not asked what area is or how they know what it is, but are simply given a formula and some numbers and asked to find the area of a triangle (see Figure 4).

Willard has a stained glass window with one triangular piece, as shown below.

[Diagram of a triangle with dimensions 6 in. and 8 in.]

What is the area, in square inches, of the triangular piece?
A 14
B 24
C 48
D 96

Figure 4. Problem 17 of the 2005 NYS mathematics sample test provides a formula and numbers to insert into the formula.

This formal way of expressing area surely makes things easier for students, but it can also take away the enriching experience of thinking at a critical, higher-order level.

Formulas and Formal Proofs

In high school, many students are pressured to memorize formulas and use equations. These formulas and equations help simplify problems and usually cut down on the time it takes to solve the problem. Although the formulas are usually easy to memorize, they cut out the
major part of understanding concepts (Battista, 1982; Brown, 1982; NYSED, 2005a; NYSED, 2005b; NYSED, 2005c; NYSED, 2005d;).

One such concept, length, can be physically measured with a ruler, whereas there is no set tool that one can use to find the area of any polygon (Battista, 1982). Even if there were such a tool, like an inch unit square, or a centimeter unit square, students would not be able to measure the areas for triangles, circles, or any non-rectangular shape. Therefore, area is calculated indirectly by using the measurements that can be found using a measuring device such as a ruler and inserting those measurements into a formula. But where did this formula come from? Many of students know the formula but could not explain where it came from or why it is formatted the way it is. One expert states the following:

One undesirable consequence of this indirect procedure for measuring area is that many students have developed extremely vague or false notions about the concept of area. Asked to define it, many of these students respond that area is “length times width.” (Battista, 1982, p. 362)

Brown (1982) agrees with Battista (1982) in the sense that students can get a better understanding of formulas by proving them true through conjectures. Brown (1982) proposes multiple questions that guide students to find a formula for simple problems that works for more complex problems. Figure 5 shows how Brown (1982) guides students into creating their own ideas and coming up with formulas.
In the question shown by Figure 5, the students are to start out with basic polygons, such as triangles and quadrilaterals, and organize their information into a chart that helps them understand the formula that follows. The formula to find the diagonals in the problem above is \( n \left(\frac{n-3}{2}\right) \) for \( n \) sides. NYSED (2005a) can pose a question like the one given in Figure 4 where the student does not need to know anything about the problem other than being able to insert numbers in place of the letters in the formula. Expecting middle school students to derive formulas from patterns may be a lofty expectation, but learning where the formula came from is very important.

Another mistake that is made is one that comes from the students’ understanding, or rather misconception, of area. A study was conducted in Ankara, Turkey where students were tested on this misconception. This study concluded that 7th grade students only have a vague understanding of perimeter and area (Şışman & Aksu, 2009). The study stated that 80% of the students who participated in the study believed that the perimeter of a figure remained the same.
if the figure was rearranged. On the other hand, only 48% of the students knew that area remained constant for a figure if its shape was rearranged (table 1).

Table 1

*Percentage of 7th Grade Students with Correct Answers*

<table>
<thead>
<tr>
<th></th>
<th>Calculate with Formulas</th>
<th>Calculate without Formulas</th>
<th>Does calculation stay the same if figure is rearranged?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perimeter</td>
<td>70.0%</td>
<td>57.5%</td>
<td>20.0%</td>
</tr>
<tr>
<td>Area</td>
<td>64.9%</td>
<td>28.4%</td>
<td>48.0%</td>
</tr>
</tbody>
</table>

These misconceptions need to be studied further, and teaching strategies need to be adjusted to correct this problem.

New York State determines if students understand the concept of area by asking an area question and providing an equation which allows for simple algebra to be mistaken for a deeper meaning (refer back to Figure 4; NYSED, 2005a). However, some students may ignore (or forget) the equation needed and use an informal technique to find the correct area of a triangle by making rectangles out of the two right triangles and ultimately proving the area formula without even knowing it. Some students can do this by tapping into a skill called spatial visualization.

**Spatial Visualization and Geometry Knowledge**

Spatial visualization is a skill set that people acquire and use to help them learn a variety of concepts. It can be used to organize and group together similar items, or visualize physical objects. Getting the grasp of different geometrical shapes can be difficult without good spatial visualization. This skill is very important when mentally manipulating figures, especially when discussing two-dimensional and three-dimensional objects. Spatial visualization can play a key
role in understanding how area formulas were derived for different polygons (Battista, 1989, 1990; Boakes & Pomona, 2009; Capraro, 2001; Clements, 1998; Gal & Linchevski, 2010; Gilchrist, 2002; McLeay, 2006).

In a mathematics sense, spatial visualization is the understanding and manipulation of two-dimensional and three-dimensional objects through mental imaging (Clements, 1998). In his study, Battista (1990) found that geometry knowledge for males and females is affected greatly by both logical reasoning and spatial visualization. Others have agreed with these findings, stating that mathematics achievement and spatial ability are related (Clements, 1998). If one can visualize a two-dimensional or three-dimensional object, it would be easier to manipulate the object by rotating it and moving it around. Since some geometric figures can be difficult to draw or represent in a lower dimension, visualizing the figures in space is crucial to understanding them.

The majority of all geometric formulas have evolved from the manipulation of pictures and diagrams which can be completed through the use of spatial visualization. To understand the area formula for a parallelogram, one must visualize or conceptualize the shape and manipulate it by creating perpendicular lines and moving pieces around to see that a rectangle can be created from a parallelogram (see Figure 6). The area formula for a rectangle is \( A = b \times h \). The breaking down and rebuilding of geometric shapes is essential to geometry (Gal & Linchevski, 2010) and understanding geometric formulas.
Figure 6. Illustration on how to manipulate a parallelogram to create a rectangle.

While students solve problems their thought process becomes complex. Student thought process needs to be studied further, because understanding the way students learn and think can lead to improved teaching methods. These improved teaching methods will in turn help future students grasp difficult concepts like area.

III. Experimental Design

The current experiment tests the hypothesis that when given a problem that compares the sizes of two different shapes, middle school students rely on memory and area formulas. The experiment consisted of a test/worksheet involving multiple problems in which the students compared the size (area) of two different shapes.

Subjects

The experiment portion of the study took place at Brocton Middle School in Brocton, NY. Brocton is located about 60 miles south of Buffalo, NY and 45 miles northeast of Erie, PA. Surrounded by farmland and vineyards, Brocton is a rural community that has a total of one school building which houses elementary, middle, and high school students. The town has a population of approximately 1,547 people. A state prison is located nearby. The school hosts
grades pre-k through twelve, and has a total of about 750 students, with 374 students at the middle and high school grade level. Of these 374 students, 27% are eligible for free lunch and 14% receive reduced-price lunch. There are two students in the school who have limited English proficiency (Wikipedia, 2011).

According to Wikipedia, a large majority of the village of Brocton is Caucasian, while there are a number of other ethnicities in the area as well. The village is 96.38% Caucasian, while 1.29% is Native American, 0.65% African American, and the rest consist of various other races. The middle and high school consists of 91% Caucasian students, 6% Hispanic or Latino, 1% Black or African American, 1% American Indian or Alaska Native, and 1% that is multiracial.

The subjects in this experiment consisted of the students in two seventh-grade mathematics classes that met every day. The classes were heterogeneously mixed according to their ability level. Most of the students have been taught mathematics in the same school district (Brocton) since they began school in kindergarten. The first class that participated ran from 8:52am – 9:32am and was made up of 18 students (17 Caucasian and one African American). Out of these 18 students, 13 were male and five were female. There were two students with IEP’s and one student with a 504 accommodation plan.

The second class that participated consisted of 24 students and ran from 9:35am – 10:15am, right after the first class that participated. In this section, the male to female ratio was 12 males and 12 females. Overall, there were 20 Caucasian students, two African American students, two Hispanic students, and a total of two students with a 504 accommodation plan. See table 2 for a breakdown of the total demographics from both classes by sex.
Table 2

*Participant Characteristics*

<table>
<thead>
<tr>
<th>Race</th>
<th>Male</th>
<th>Female</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Caucasian</td>
<td>27</td>
<td>10</td>
<td>37</td>
</tr>
<tr>
<td>African American</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Hispanic</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>29</td>
<td>13</td>
<td>42</td>
</tr>
</tbody>
</table>

The seventh grade level was chosen because it is at the middle school level that is believed by the researcher’s colleagues and their mentoring professors to be a transitioning period. The transitioning is believed to be occurring in the students’ thought processes. Specifically, this is the transition from using pictures and shape manipulation to the more formal way of proving area, which consists of using formulas and substituting numbers for variables.

**Design**

The experiment was completed in two different, 40-minute seventh grade class periods during the same day. During class, the students were given a worksheet consisting of nine questions that asked the students to answer which of two shapes was larger, or to indicate that they were the same size. After answering the question, they then justified their answer in writing. Two questions regarding the students’ knowledge of area and perimeter were included at the end of the original nine questions. Students were given room on their worksheets to perform calculations, draw diagrams, and explain their answers.

Before the worksheet was administered, the instructor clarified what was meant by the term ‘larger’ or ‘bigger,’ so that the students who could have experienced difficulties figuring it out were able to understand the idea of the problems. What the instructor said to the students
was critical and had the ability to affect the students’ answers and problem-solving techniques. In order to stay unbiased, the instructor used the following statement: “In order to compare how large a shape is, you need to compare the amount of space, or area, that the shape encompasses.” This statement should not have pushed the students in either direction in solving the problems, but provided them with instructions on how to complete the worksheet (see the worksheet attached in the Appendix A).

IV. Methods of Analysis

After the students completed the worksheet, worksheets were collected and analyzed. During the analysis, two major concepts were examined. The first concept was the type of strategy that the student used on the worksheet. There were four major categories for the types of strategies that were used to solve these problems: formal, informal, visual, or no answer.

The formal strategy of solving these problems consisted of the students examining the measurements and using a formula to determine the areas of the given figures, then comparing the numerical size of the shapes. The other possible way of addressing the size problems consisted of making a judgment from spatial visualization or “manipulating” the shapes. This manipulation involved strategies such as re-drawing the shapes, re-arranging a shape to look like another, or even manipulating them abstractly in their heads with their own spatial visualization. The response was placed in the “no answer” category if the student left the question blank. If a student answers a question but gives no indication to what strategy was used, if any, then the answer was placed in the visual category. Any other strategy not mentioned here was placed into one of the four categories (see Appendix B for grading and categorizing rubric). Categorizing the explanations was completed in both a qualitative and quantitative manner. A student may have
used formulas in their head but did not write an explanation or enough of an explanation to give indication of formula use. The explanation would then be qualitatively judged for information, or lack of information, then quantitatively categorized in one of the four categories.

The second major concept that was examined in this experiment was the correctness of the given answers. Grading the correctness of the problems satisfies the second part of the hypothesis and determines which strategy was more accurate at answering the questions. Some students used both types of strategies. That is why the analysis of problem-solving strategies and correctness was based on each individual problem and not on the overall worksheet. The correctness was graded in a quantitative manner with only two categories for each problem (correct, incorrect). If the student indicated the right answer, it was correct. If the student indicated a wrong answer or left no indication of an answer, it was marked incorrect.

There were four different types of questions presented in the worksheet. The first type of question involved the two shapes, whether similar or different, with lengths of sides and heights given numerically (see figure 7).

![Figure 7](image)

*Figure 7.* Question 4 from the student worksheet is an example of a straightforward question.
This type of question was chosen because it is a straightforward question that the students are familiar with. Formulas are normally provided for these types of questions on a typical homework assignment or assessment.

The second type of question chosen was similar to the first type of question, only without numbers. The shapes had measured lengths of unknown size. In Figure 8 below, the unknown size is of length “x”. This question provides the students with necessary information on the size of the shapes but does not directly influence the student to use a formula or any other particular method.

![Figure 8](image)

*Figure 8.* Question 2 from the student worksheet is an example of defining lengths without numbers.

The third type of problem displays shapes on a grid. This allows for the students to count the squares on the inside, determine the length of the sides, or easily rearrange the shape without losing the integrity of the shape. Figure 9 provides an example of this type of problem.
Figure 9. Question 5 from the student worksheet is an example of shapes on a grid.

Finally, since numbers tend to influence students to use formulas, a question was developed to counteract the first type of question that gave lengths of sides. Figure 10 was included in the worksheet to challenge the students to use methods other than formulas to solve the problem at hand. This question shows a rhombus and a square with hash marks indicating that all the sides are equal.

Figure 10. Question 9 on the student worksheet provides non-numerical lengths of sides.

The analysis of the data either supported or went against the hypothesis that middle school students use formulas more often and that the use of formulas is not as accurate as knowing the informal way of solving area problems.
V. RESULTS

This study produced both major and minor results about students' geometric thought process. The results of this study display which problem solving technique is utilized the most, which problem solving technique was the most accurate, and the students' overall perception of area.

Overall Results

- The use of formulas exceeded the use of informal problem solving throughout the geometry worksheet.
- The students were correct more often when they used an informal approach to solving the question, than with a formula or any other method.
- Gender was a factor in method used but not in correctness.
- There is not enough evidence to suggest that class section was a factor in method used or in correctness.

- Question number 8 was the most difficult question for the students:

Worksheet results

The results from the student worksheet displayed evidence that supports the hypothesis. After the correction of the worksheet it was determined that students do rely too much on formulas and when given questions without formulas, the students would not be able to answer them with great precision. A 0.05 level of significance was used in the analysis of variance to determine the following results.
• The use of formulas exceeded the use of informal problem solving throughout the geometry worksheet: There were 148 total questions answered using formulas as compared to only 86 questions answered using an informal approach.

Table 3

Problem Solving Approaches

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Informal</th>
<th>Visual or No Reason</th>
<th>No Answer</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>148</td>
<td>86</td>
<td>137</td>
<td>7</td>
<td>378</td>
</tr>
</tbody>
</table>

There was a high number of responses on the worksheet that had no reasoning for their answer and were classified as “visual or no reason” as the approach used. There were also a total of 7 questions that were left blank out of the sample set.

• The students were correct more often when they used an informal approach to solving the question, than with a formula or any other method: While using an informal approach, the students were correct 82.6% of the time, as compared to using a formula, the students were only correct 63.5% of the time. Table 3 below shows the comparison of the formal and informal approaches with regard to correctness.
Table 4

*Thought Process by Correctness Interaction*

<table>
<thead>
<tr>
<th>Thought Process</th>
<th>Correct</th>
<th>Incorrect</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correctness</td>
<td>Formal</td>
<td>Informal</td>
<td>Total</td>
</tr>
<tr>
<td>Correct</td>
<td>94 (63.5%)</td>
<td>71 (82.6%)</td>
<td>165</td>
</tr>
<tr>
<td>Incorrect</td>
<td>54 (36.5%)</td>
<td>15 (17.4%)</td>
<td>69</td>
</tr>
<tr>
<td>Total</td>
<td>148 (100%)</td>
<td>86 (100%)</td>
<td>234</td>
</tr>
</tbody>
</table>

The bar chart (Figure 11) below shows the percent correct for each method, of all the answers given (ex. for all of the responses that were answered using a visual approach, about 55.5% were correct).

*Figure 11.* A bar graph illustrating Correctness by Method.

Only 55.5% of the questions that were answered using the visual method were correct (44.5% were incorrect). The students were 82.6% correct (17.4%
incorrect) while using an informal approach, and only 63.5% correct (36.5% incorrect) when using a formula. None of the answers were correct (100% incorrect) when students left the answer blank.

- **Gender was a factor in method used but not in correctness:** Gender did not play a role in whether the student used a formula or left the answer blank (p-values of 0.762 and 0.442, respectively). The males answered using formulas about 40% of the time as compared to the females at 38%. The females did not answer about 1% of the time while the males did not answer about 2% of the time. However, with a p-value of 0.000 and 0.011, it was determined that the use of informal methods and the visual or no reason method were quite different for males and females. The males answered using an informal method about 16% of the time, as compared to the females at 31%. From the data it was determined that the females seemed to provide justification for their answers more frequently than the males because the visual or no answer method was used about 44% of the time for the males and only 30% for the females.

  A p-value of 0.219 shows that gender is not a factor in the model predicting correctness.

- **There is not enough evidence to suggest that class section was a factor in method used or in correctness:** There was not enough evidence to prove class section had an effect on the methods chosen to use or the number of correct answers per student. P-values ranged from 0.366 to 0.984 when calculating a one–way analysis of variance of class time predicting methods used, therefore it was rejected as a possible predictor.
Class section was also rejected as a predictor for the number of correct answers per student, with a p-value of 0.567.

- **Question number 8 was the most difficult question for the students:** The questions varied in difficulty and information provided, and question number 8 was answered incorrectly more than any of the other questions. Approximately 84.3% of the students answered question 8 incorrectly. The line graph (Figure 12) below shows the percentage correct for each method and for each question.

![Line Graph](image)

*Figure 12.* An illustration of correctness by both method and by question using a line graph.

The line graphs represent the percent correct by each method and percent correct by each question. The horizontal line in the graph shows where 50% is located. From the graph it is known that the informal method of problem solving was the correct the most often and the plot for it is located about midway between
80% and 90%. It is also true that question number two was answered most accurately (marginal amount above 70%).

The students used different methods to solve the problems and therefore approached this question differently. Even though question 8 was later in the worksheet, where the students may be less enthusiastic about getting the right answer, it is believed that this question was answered incorrectly more because it includes a shape that the students may not have had much practice with previously. The trapezoid may not be the only factor in this question, because the students were not given any measurements of the trapezoid since it was displayed on a graph where the students could count squares inside the shape or find out the measurements. Figure 13 shows question number 8 from the worksheet.

![Figure 13. Question number 8 from the student worksheet has a trapezoid on a grid.](image)

**Anomalies**

Along with question number 8 being significantly different from the other questions, some of the students' answers are exemplary. Some answers showed ideal explanations, while other answers were abstractly justified.
Figures 14 and 15 show students’ work where formulas were attempted but incorrectly used.

Figure 14. A student incorrectly uses formulas and obtains a correct answer.

Figure 15. A student also incorrectly applies formulas to the figures here but acquires the correct answer.

Since students did not have a list of formulas for them to refer to they could easily confuse area and perimeter formulas if trying to recall from their memory. The student in Figure 14 displays some knowledge of area for both shapes but forgets to square the radius when finding the area of the circle. Not knowing the whole formula for area of a circle has the student
in Figure 14 producing incorrect measurements for the circle. Figure 15 shows a student attempting to use formulas but confuses circumference of a circle with area of a circle and uses algebra on familiar numbers. Some students came up with the correct answers even though their justification is wrong.

Figures 16 and 17 both show the same question and informal approaches to that question. The student from Figure 16 has correctly applied an informal method and either knows where the formula for trapezoid is derived from or would easily grasp the explanation of the formula for a trapezoid. The student from Figure 17 either displays some knowledge of how to informally solve the problem of the trapezoid and improperly applies it to this problem, or just wants the two shapes to be the same so they can compare more easily.

Figure 16. A student has correctly applied an informal approach to question 8.
Figure 17. A student has incorrectly applied an informal method to question 8.

The common misconceptions on this worksheet either involved a formula confusion like in Figures 14 and 15 or an improper application of an informal approach like Figure 17. Other misconceptions on this worksheet include examining the two shapes and stating that one side looks longer and therefore the figure has to be larger.

**Area vs. Perimeter**

The worksheet included two questions pertaining to area and perimeter and their relation to each other. Questions 10 and 11 were used to get a better insight of the students’ grasp of the perimeter concept as compared to the area concept.

Question 10 presented the students with a triangle with lengths of the sides given and asked the students if there was a way to create another triangle that had a larger perimeter but
smaller area. Only 9 of the 42 (21.4%) students correctly answered that a triangle was possible to create. Those students displayed a thin triangle with two long sides and one short side.

Question 11 was a maple leaf figure on a grid with a grid key given. This question asked the students to find both the perimeter and the area of the figure. While the perimeter integrated a more complex concept with some of the sides of the figure drawn diagonally across a unit square giving it a length of $\sqrt{2}$, it was believed to be within the students’ range of knowledge. However, the results proved this statement to be incorrect. None of the students correctly answered the perimeter question. On the other hand, the majority of the students were close by giving the diagonal a length of 1 and adding the sides.

The area of the figure in question 11 was only correctly calculated by three students. Since there is no area formula established for this type of shape, most of the students were stumped and did not answer, while others guessed. The three students that answered correctly divided the large shape into sections and smaller more manageable shapes like squares and triangles. Figure 18 shows how one of three students solved the area dilemma.

![Figure 18](image_url)
Overall, formulas and formal problem solving techniques were chosen more often than an informal approach. However, the informal approaches proved to be more precise and accurate. In general the students did not have a complete understanding of the concept of area at the time of this study.

VI. Implications for Teaching

This study was implemented to gain a better understanding of how middle school students think about and solve area problems. The hypothesis predicted that students would rely on formulas when they could, and that the students’ ability to use formulas is less accurate than other methods of solving area problems. Based on this study’s results, formulas are actually hindrances, rather than the helpful tools most people think they are. These hindrances can evolve into a useful crutch with proper instruction. Below are discussed four suggestions for implementation that emerged as a result of this study.

**Concept Introductions are Crucial**

The overuse of formulas could possibly stem from how the students were introduced to the concept of area. If formulas were used from day one in geometry, the students would be more likely to fall back on formulas because that is what they know and trust. The first way a student’s understanding of area can be strengthened is to introduce the concept without any formulas. Introducing area by displaying regular shapes on a unit grid and explaining how area is the space within the 2-dimensional objects is a great way to create a foundation for understanding. Making the students count the number of squares that lay within the boundaries of a shape helps them understand that area is the space that the shape encompasses.
**Stray from the Norm and Estimate**

Students may initially experience anxiety in the attempt to find the area of irregular shapes. However, this can become a simple task if the irregular shapes are broken down into smaller parts or rearranged. This technique can help students strengthen their understanding and prepare them if they come across any unknown shapes.

The use of estimation can come in handy when using irregular shapes on a grid. If students understand that area is the space within the interior of a figure, then students can count the squares in the shape. If the sides of an irregular shape cut through the squares on the grid in a unique fashion, then estimating the number of squares within the shape will give an approximation of the area. To do this, one would count the number of full squares inside the boundaries of the shape and add it to an estimated area of the squares that are cut off. A good way to estimate the area of the squares being cut off is to count the number of squares that are intersected and divide that number by two. Adding the two numbers together gives a quick and fairly accurate estimation of the shape’s actual area.

**Defend and Develop Formulas**

While knowing the area formula for a shape is important, knowing where the formula came from and how it was developed is more important. If a student knows how the area formula was developed, then they will not need to memorize formulas because they will be able to develop them on their own. The development and defense of the formulas is a strategy that can be implemented after the students have been introduced to area. It can even serve as an intervention strategy if a student already has a misconception of area.
Having the student(s) understand that area is the space within the borders of a shape is crucial in order to develop a formula for any shape. The development of the area formulas also needs to have a certain order. The basic shapes of the square and rectangle need definitive formulas before one can move on to defining the area of a parallelogram or a trapezoid. The parallelogram and trapezoid can then be defined by rearranging parts and creating a rectangle.

**Integrate Informal Approaches**

The use of informal problem solving is not just a technique for solving area problems but can be used in many different areas across the mathematics curricula. Integrating informal problem solving techniques can be difficult in algebra, trigonometry, or even in statistics but can help the students develop their own techniques for geometry and get them to think “outside the box”. The added practice with informal approaches in the other areas of mathematics will train the students to use informal strategies when solving problems they do not understand.

**Suggestions for Future Research**

Even though this research provided supporting evidence of the thought processes of middle school students, there are some questions left to be solved. The original belief that younger students rely more on informal approaches may still be true, but what about high school students? Do high school students only use formulas because that is what they are used to or are they dynamic and can use informal approaches? Mathematics professionals are another group of subjects that could provide interesting data on their accuracy and use of methods when comparing areas. Without any previous instruction, the way students solve area problems of irregular shapes would be another interesting topic to investigate.
Concluding Remarks

This study was conducted to gain a better understanding of middle school students’ problem solving techniques with regard to 2-dimensional shapes. The study was also conducted to develop strategies to help current and future students in their mathematical development. Based on the results of the study, middle school students do not have a thorough understanding of the concept of area and have a tendency to use formulas more than other strategies to solve area problems. It was also concluded that informal methods of solving area problems produced more correct responses.
VII. References


Appendix A

Data Collection Worksheet

Gender: Male Female

Which shape is larger?

Indicate which shape is larger or if they are the same size. Justify your answer using the diagram and space below. In each question there is a red and blue figure (green indicates shared sides), identify which one is larger or if they are the same size. Show all your work.

1)

![Diagram of shapes 1](image1)

2)

![Diagram of shapes 2](image2)
5) 

6)
9) Drawn to scale

10) Can a triangle be created that has a larger perimeter but smaller area than the one below? If yes, show an example and explain. If not, explain why not.
11) Find the area and perimeter of the shape below. (The distance between the dots horizontally and vertically is 1.)

Area = 
Perimeter =
Appendix B

Scoring Rubric

Correctness

<table>
<thead>
<tr>
<th>Correct</th>
<th>Incorrect</th>
</tr>
</thead>
<tbody>
<tr>
<td>(correct shape identified as being larger)</td>
<td></td>
</tr>
<tr>
<td>- Correct color and/or shape are indicated as being larger</td>
<td>- Incorrect color and/or shape indicated as being larger</td>
</tr>
<tr>
<td>- Correct shape is circled.</td>
<td>- Incorrect shape circle</td>
</tr>
<tr>
<td>- Formulas written to indicate correct shape is larger than the other, numerically.</td>
<td>- Formulas written to indicate incorrect shape is larger</td>
</tr>
<tr>
<td>- Indicating both shapes are of equal size when they are.</td>
<td>- No answer</td>
</tr>
<tr>
<td>- Informal procedures lead to indication of correct shape.</td>
<td>- Answering one shape is larger when they are the same size</td>
</tr>
<tr>
<td></td>
<td>- Informal procedures lead to incorrect shape.</td>
</tr>
</tbody>
</table>

Strategy

<table>
<thead>
<tr>
<th>Formal</th>
<th>Informal</th>
<th>Visual</th>
<th>No Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>- The sole use of formulas to answer the question</td>
<td>- Use of method that was not categorized as formal or visual</td>
<td>- Indicated or stated answer with no reason or explanation</td>
<td>- Left question completely blank</td>
</tr>
<tr>
<td></td>
<td>- Manipulation of shapes to compare more easily.</td>
<td></td>
<td>- Explanation stated the student just looked at the shapes</td>
</tr>
</tbody>
</table>
Appendix C

Parent Consent Form

TO: Parents/Guardians of Students in 7th grade Mathematics

From: Reid Bland

DATE:

RE: Consent form

Purpose, Procedure, and Benefits

➢ The purpose of this study is to better understand the thought process of middle school students when solving area problems.

➢ This study will consist of 1 worksheet with 6 area problems on it, asking the students to compare the sizes of two shapes with measurements given and justify their answers. The data collection for this study will only last one 40 minute period.

➢ The goal of this study to understand the thought process and develop more effective ways of teaching the concept of area. Not only can your students benefit from the developed teaching methods, but many other students across the mathematics spectrum.

Related information

➢ Your student has been asked to participate in this study because they are a member of Mr. Williams' 7th grade math class.

➢ To ensure confidentiality neither your name nor the name of your student will be used in this study. The students will not put their names on the worksheet and only the primary investigator
(Mr. Reid Bland) and his college advisor (Dr. Keary Howard) will have access to the completed worksheets.

- There is no cost (nor compensation) for participation in this study.
- The student may withdraw from the study at any point if they feel uncomfortable participating. The students may also speak with the primary investigator or their teacher, Mr. Williams, (if they feel more comfortable) at any time if they have questions about the worksheet or the overall investigation.
- Only minimal risks (if any) are anticipated for the students. The worksheet has no right or wrong answers making this study pressure free.
- Please read over and discuss the information with your student to make you both are fully aware of everything involved in this study.
- For additional information or questions, please feel free to contact Mr. Bland by email:
  blan3316@fredonia.edu or by phone: 716-969-5564
- You may also contact Mr. Bland’s college advisor, Dr. Keary Howard, at SUNY Fredonia by email:
  keary.howard@fredonia.edu or by phone: 716-673-3873

Please sign the attached consent form and return it with your student. This consent form authorizes the use of data from your student’s completed worksheet for the purpose of research. Thank you in advance for your time and consideration.
Parental Consent Form

SUNY Fredonia

Please print and sign your name below to indicate permission to collect data from your student’s worksheet. Feel free to keep a copy of the consent form for your files. Your participation is important and is appreciated greatly.

**Voluntary Consent:** I have read this memo and agree to allow my son/daughter to participate in this study. If I withdraw my son/daughter from the study, I understand that there will be no penalty placed upon him/her. I understand that my son’s/daughter’s confidentiality will be maintained. I understand that if I have any questions regarding this study, I may contact Reid Bland at 716-969-5564 or at blan3316@fredonia.edu

Please return this original, completed consent form as soon as possible. Thank you for your cooperation.

Parent/Guardian Name (please print) ____________________________________________

Parent/Guardian Signature ____________________________________

Date: ____________________
Appendix D

Student Consent Form

TO: Students in 7\textsuperscript{th} grade Mathematics

From: Mr. Bland

DATE:

RE: Consent form

 önemlilik etmek zorunda oldum ya da zorunlu olmadan, çalışmaya katılmak ve katılmamak benim notlarımıza her iki durumda da etkisel olmaz.

To participate, you need to simply get the permission forms signed and complete 1 worksheet.

The worksheet will not affect your grade.

Your answers on the worksheet will be the main focus of this study.

Your name will not be used in any way and the study will remain confidential.

Examples of your work may be included as a part of the project, but no names or identities will be given.

The risks for you in this study are very small and there may be no risk at all to you. However, if at any point you feel uncomfortable or there is a problem that you would like to discuss with someone other than Mr. Bland, your teacher Mr. Williams will be in the classroom and aware of the study that is going on.

Please remember that this study will be used to make learning easier for you and other future students.

There is no penalty for not signing the consent form.

You will not be paid or given rewards for your participation.

Please discuss this with your parent or guardian. If you have any questions feel free to ask.
Student Consent Form

SUNY Fredonia

Thank you for being a part of this study. Please print and sign your name below to indicate permission to collect data from your student’s worksheet. Feel free to keep a copy of the consent form for your files. Your participation is important and is appreciated greatly.

**Voluntary Consent:** I have read this memo and agree to participate in this study. I have discussed this memo with my parent or guardian. If I withdraw from the study, I understand that there will be no penalty placed upon me. I understand that my confidentiality will be maintained. I understand that if I have any questions regarding this study, I may contact Mr. Bland at 716-969-5564 or at blan3316@fredonia.edu

Please return this original, completed consent form as soon as possible. Thank you for your cooperation.

Student Name (please print): ____________________________

Student Signature: ____________________________

Date: __________________

Parent/Guardian Signature (witness): ____________________________

Date: __________________