Maximal Foliations in Spacetimes with Spacelike Translational Symmetry

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Andrew Bulawa

We, the dissertation committee for the above candidate

for the Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Michael Anderson - Advisor
Professor, Department of Mathematics

C. Denson Hill - Chairperson of Defense
Professor, Department of Mathematics

Marcus Khuri
Assistant Professor, Department of Mathematics

Martin Roček
Professor, Department of Physics

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

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Foliations of spacetime by maximal (mean curvature zero) spacelike hypersurfaces have long been studied in relativity. They allow one to reduce the Einstein equations to a system of hyperbolic evolution equations coupled with an elliptic lapse equation. This simplification made possible the celebrated stability result of Christodoulou and Klainerman, where maximal foliations were used to show that asymptotically flat Cauchy data which is sufficiently small gives rise to a complete globally hyperbolic spacetime.

This thesis considers maximal foliations in spacetimes which admit translational symmetry in the form of a nonvanishing spacelike Killing field. In this setting, it is generally conjectured that noncompact symmetric Cauchy data with appropriate asymptotics gives rise to a complete globally hyperbolic spacetime. While proving the conjecture is beyond the scope of this work, properties of maximal foliations in the symmetric setting are studied, and their utility towards proving this conjecture is discussed.

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Chapter 1

Introduction

The analysis of spacetimes $(M, \bar{g})$ satisfying the Einstein vacuum equations $Ric_{\bar{g}} = 0$ is far from being straightforward. This is due primarily to the fact that the Einstein equations in general coordinates form a highly nonlinear system of partial differential equations of no particular type. Fortunately there are ways to formulate the equations in a more tractable form.

The wavelike gauge, or harmonic gauge, was the first procedure to achieve significant success for an initial value formulation. This method involves generating coordinates with respect to which the Einstein equations reduce to a system of hyperbolic equations. Choquet-Bruhat used this gauge to prove local existence and uniqueness for the Cauchy problem [9]. Choquet-Bruhat and Geroch were able to extend this to existence and uniqueness for inextendible Cauchy developments [10]. However, these results did not give control of the asymptotic behavior of the solutions. Success along these lines was achieved by Christodoulou and Klainerman in their celebrated proof of the stability of Minkowski space [11]. There a different gauge was used, the maximal hypersurface gauge.

The maximal hypersurface gauge involves a foliation of spacetime by spacelike hypersurfaces each having mean curvature identically zero. These are referred to as maximal hypersurfaces in the Lorentzian setting. In this framework, solving the Einstein equations may be reduced to solving a reduced system of quasilinear hyperbolic evolution equations coupled with an elliptic equation for a lapse function u on the hypersurfaces.

Christodoulou and Klainerman were able to take advantage of this simplification to establish geodesic completeness along with asymptotic estimates for the evolution of suitably small and asymptotically flat Cauchy data. While Lindblad and Rodnianski were able to prove stability of Minkowski space in the wavelike gauge, which came as a surprise given the suspected limitations of that procedure [19], their proof does not provide the same level of asymptotic control. It is in this way that the maximal hypersurface gauge stands apart from other techniques, and this is the motivation for considering it here.

Consider Cauchy data (Σ, g, k) for the Einstein equations consisting of a noncompact complete three-manifold, a Riemannian metric g, and a symmetric two-tensor k. Suppose that the data is invariant with respect free S^1 action. It is generally conjectured that such data, subject perhaps to suitable smallness conditions and fall-off at infinity, will give rise to a geodesically complete spacetime development M.

The symmetry introduces some simplifications, but it carries with it some difficulties as well. An obvious simplification arises from the fact that the analysis may be performed on the manifolds obtained as quotients with respect to the symmetry. The vacuum equations on the four-dimensional spacetime M take the form of gravity coupled with scalar fields representing the geometry of the S^1 bundle over the three-dimensional quotient spacetime. Convenient examples in this class of spacetimes are given by the Einstein-Rosen waves discussed in Section 3.3. These are particularly attractive due to the fact that they are determined by solving the linear wave equation on a flat Minkowski space.

This first main result, Theorem 3.4, addresses the issue of asymptotic behavior of the lapse function. For the maximal gauge in the nonsymmetric setting, one assumes that the lapse function on asymptotically flat Cauchy data is asymptotic to a constant. Theorem 3.4 asserts that, for S^1-symmetric data, a bounded lapse function on a maximal hypersurface must be constant and forces the hypersurface to be totally geodesic. This illustrates that the maximal hypersurface gauge with a bounded lapse can only give rise to a static spacetime development, which can be complete only if it is the flat Minkowski space, or a quotient thereof.

This brings forth a question of what asymptotic behavior is appropriate for the lapse function. Some control of the lapse function is apparent by the first assertion in Lemma 3.6.

An issue regarding the asymptotic geometry of Cauchy data is that of multiple ends. These are unbounded components of the data which remain after the removal of a compact set. It is on each of these ends where the asymptotic flatness conditions are to be imposed. Theorem 3.7 shows that, in the case of S^1-symmetric Cauchy data, it is only necessary to consider a single end. It is also established that the surface obtained as a quotient of nontrivial data is conformally equivalent to the Euclidean plane.

A difficulty presented by the symmetry is its incompatibility with the traditional notion of asymptotic flatness. Efforts have been made (c.f. [4], [5]) to find a definition of asymptotic decay which accommodates the symmetry. The most straightforward approach is to impose a condition on the two-dimensional Cauchy data in the spacetime quotient. There it is found that instead of requiring that the metric asymptotically approach Euclidean space, as in the nonsymmetric case, it is more natural to ask that it is asymptotic to a Euclidean cone. In the case of Einstein-Rosen waves, this is exactly what happens. The asymptotic cone angle is determined by the energy of the solution to the linear wave equation which determines the spacetime. It may be that a bound on a quantity analogous to this energy will give similar asymptotics for general S^1 symmetric data. This is discussed at the end of Chapter 3 and partially addressed by Theorem 3.8.

Chapter 2

Maximal Gauge and the Lapse Equation

Let $(M, \bar{g})$ be a smooth four-dimensional manifold with a smooth Lorentzian metric $\bar{g}$ of signature $(-,+,+,+)$. The induced connection is denoted by $\bar{D}$, the Ricci curvature by $\bar{R}ic$, and the scalar curvature by $\bar{R}$.

The vacuum Einstein equations for $(M, \bar{g})$ are given by

$$\bar{R}ic = 0. \tag{2.1}$$

When (2.1) is satisfied, $(M, \bar{g})$ is called a vacuum spacetime.

Spacetimes of interest here are globally hyperbolic (see [18] or [21]). These spacetimes admit a global time function, the level sets of which form a foliation of M by spacelike hypersurfaces. In the next section, some basic geometric structures relating the geometry of hypersurfaces to that of the spacetime are defined. In section 2.2, maximal hypersurfaces are defined and the stability operator is introduced. Finally, maximal foliations and the lapse equation are discussed in section 2.3.

2.1 Hypersurface Geometry

Let Σ be a hypersurface in M. At points in Σ, the tangent space TM of M may be viewed as a vector bundle over Σ which splits as $TM = T\Sigma \oplus T\Sigma^{\perp}$, where $T\Sigma^{\perp}$ represents the normal bundle over Σ.

A vector X in TM is called timelike, spacelike or null provided its square length $\bar{g}(X, X)$ is nonpositive, nonnegative or zero, respectively. The hypersurface $\Sigma \subset M$ is called spacelike if $T\Sigma^{\perp}$ consists entirely of timelike vectors.

Assume that Σ is spacelike. Then Σ inherits a Riemannian metric $\bar{h}$, obtained by restricting the action of $\bar{g}$ to vectors in $T\Sigma$. Σ also inherits a connection $\bar{\nabla}$ given by

$$\bar{\nabla}_x y = (\bar{D}_X Y)^T,$$

for $x, y \in T\Sigma$ and where ^{T} denotes the orthogonal projection $TM \to T\Sigma$, and X and Y are arbitrary extensions of the vector fields x and y to a neighborhood of Σ in TM. The definition is independent on the choice of extensions. Therefore, $\bar{\nabla}$ agrees with the canonical connection given by $\bar{h}$.

Suppose that Σ is orientable. Then a global section $\bar{\nu}$ of $T\Sigma^{\perp}$ may be chosen with unit length, i.e. $\bar{g}(\bar{\nu}, \bar{\nu}) = -1$. The second fundamental form $\bar{k} : T\Sigma \otimes T\Sigma \to \mathbb{R}$ associated to Σ is defined by

$$\bar{k}(x,y) = \bar{g}(\bar{D}_X Y, \bar{\nu}),$$

From the relation,

$$\bar{g}(\bar{D}_X Y, \bar{\nu}) = X\bar{g}(Y, \bar{\nu}) - \bar{g}(Y, \bar{D}_X \bar{\nu}) = -\bar{g}(Y, \bar{D}_X \bar{\nu}),$$

one verifies that $\bar{k}$ is tensorial. In addition, near any point p, one may extend vectors x_p and y_p to vector fields X and Y in such a way that $\mathcal{L}_X Y = 0$. This shows that $\bar{k}$ is symmetric.

It will be useful at times to relate the second fundamental form to the Lie derivative of the metric.

$$\bar{k} = -\frac{1}{2}\mathcal{L}_{\bar{\nu}}\bar{g} \tag{2.2}$$

This follows from the computation

$$\begin{aligned}
\mathcal{L}_{\bar{\nu}}\bar{g}(X,Y) &= \bar{\nu}\bar{g}(X,Y) - \bar{g}(\mathcal{L}_{\bar{\nu}}X, Y) - \bar{g}(X, \mathcal{L}_{\bar{\nu}}Y) \\
&= \bar{g}(\bar{D}_X \bar{\nu}, Y) + \bar{g}(X, \bar{D}_Y \bar{\nu}) \\
&= -2\,\bar{k}(X,Y),
\end{aligned}$$

where the second line is due to the compatibility of the metric with the connection and the third from the discussion above.

The Gauss equation for hypersurfaces allows one to use the second fundamental to relate the scalar curvature $\bar{s}$ of Σ to the scalar curvature $\bar{R}$ of M

by

$$\bar{s} - |\bar{k}|^2 + (\operatorname{tr}\bar{k})^2 = \bar{R} + 2\bar{R}ic(\bar{\nu}, \bar{\nu}). \tag{2.3}$$

Also useful is the Codazzi-Mainardi equation which gives

$$d(\operatorname{tr}\bar{k}) - \operatorname{div}\bar{k} = \bar{R}ic(\bar{\nu}, \cdot). \tag{2.4}$$

The trace $tr\bar{k}$ and square norm $|\bar{k}|^2$ are given at each point $p \in \Sigma$ in terms of an orthonormal basis $\{E_1, E_2, E_3\}$ of $T_p\Sigma$ by

$$\begin{aligned}
\operatorname{tr}\bar{k} &= \sum_{i=1}^{3} \bar{k}(E_i, E_i), \\
|\bar{k}|^2 &= \sum_{i=1}^{3}\sum_{j=1}^{3} \bar{k}(E_i, E_j) \cdot \bar{k}(E_j, E_i), \\
\operatorname{div}\bar{k}(\cdot) &= \sum_{i=1}^{3} \bar{\nabla}_{E_i}\bar{k}(E_i, \cdot).
\end{aligned}$$

Given a tensor $T : TM \otimes TM \to \mathbb{R}$, its trace at a point $p \in \Sigma$ may be computed in terms of a orthonormal basis $\{E_1, E_2, E_3\}$ of $T\Sigma$ and $\bar{\nu}$ by

$$\operatorname{tr} T = -T(\bar{\nu}, \bar{\nu}) + \sum_{i=1}^{3} T(E_i, E_i).$$

The minus sign arises from the fact that $\bar{\nu}$ is timelike.

One also has the following formula for the square norm of a one-form ω on M at $p \in \Sigma$,

$$
\begin{aligned}
|\omega|_M^2 &= -\omega(\bar{\nu}) \cdot \omega(\bar{\nu}) + \sum_{i=1}^{3} \omega(E_i) \cdot \omega(E_i) \\
&= -\omega(\bar{\nu}) \cdot \omega(\bar{\nu}) + |\omega|_\Sigma^2.
\end{aligned}
$$

The d'Alembertian operator $\bar{\Box}$ on M is given by its action on a smooth function $f : M \to \mathbb{R}$ by the trace of the Hessian of f on M:

$$\bar{\Box}\, f = \mathrm{tr}\, \bar{D} df.$$

One also has the Laplace-Beltrami operator $\bar{\Delta}$ for smooth functions $u : \Sigma \to \mathbb{R}$ given by the trace of the Hessian of u on Σ,

$$\bar{\Delta} u = \mathrm{tr}\, \bar{\nabla} du.$$

The two operators are related by

$$\bar{\Delta} f = \bar{\Box}\, f + \bar{D} df(\bar{\nu}, \bar{\nu}) - \mathrm{tr} \bar{k} \cdot df(\bar{\nu}), \tag{2.5}$$

where on the left, f is taken as its restriction to Σ. This follows from

$$
\begin{aligned}
\bar{\Box}\, f + \bar{D}df(\bar{\nu},\bar{\nu}) &= \sum_{i=1}^{3} \bar{D}df(E_i,E_i) \\
&= \sum_{i=1}^{3} E_i df(E_i) - df(\bar{D}_{E_i}E_i) \\
&= \sum_{i=1}^{3} E_i df(E_i) - df(\bar{\nabla}_{E_i}E_i - \bar{g}(\bar{D}_{E_i}E_i,\bar{\nu})\bar{\nu}) \\
&= \sum_{i=1}^{3} \bar{\nabla}df(E_i,E_i) + \sum_{i=1}^{3} \bar{g}(\bar{D}_{E_i}E_i,\bar{\nu}) \cdot df(\bar{\nu}) \\
&= \bar{\Delta} f + \operatorname{tr} \bar{k} \cdot df(\bar{\nu}).
\end{aligned}
$$

2.2 The Stability Operator

In the maximal hypersurface gauge, the vacuum equations (2.1) are viewed relative to a foliation of spacetime by spacelike hypersurfaces with mean curvature identically zero. These are called maximal hypersurfaces.

Definition 2.1. *A hypersurface of* $(M,\bar{g})$ *given by data* $(\Sigma,\bar{h},\bar{k})$*, where* $\bar{h}$ *denotes the induced metric and* $\bar{k}$ *denotes the second fundamental form, is called* maximal *provided* $\operatorname{tr}\bar{k}$ *vanishes identically on* Σ*.*

The nature of the foliation near a given hypersurface is described by the lapse function u, discussed in the next section, which solves a differential equation given by the stability operator L which is defined below.

Maximal hypersurfaces are the analogue of minimal hypersurfaces of Riemmanian manifolds. They correspond to critical points of volume with respect to smooth variations. To see this, consider a spacelike hypersurface $\Sigma \subset M$ with induced metric $\bar{h}$. Let $\bar{\nu}$ be a unit length vector field defined in a neighborhood of Σ in M which restricts to a section of $T\Sigma^{\perp}$ along Σ. Multiply $\bar{\nu}$ by a smooth nonnegative function of compact support to obtain a smooth vector field η compactly supported in M. Let U denote the intersection of the support of η with Σ. Suppose that U is not empty.

Let $\phi_t : M \rightarrow M$ denote the one parameter family of diffeomorphisms corresponding to the flow of η, with ϕ_0 equal to the identity. Then for small t, $\Sigma_t = \phi_t(\Sigma)$ is a smooth hypersurface of M. We may pull back by ϕ_t the volume form which $\bar{g}$ induces on Σ_t to obtain a volume form $d\mu_t$ on Σ. Integrating this volume form over $U \subset \Sigma$ defines the functional

$$\mathcal{V}(t) = \int_U d\mu_t, \tag{2.6}$$

which gives the volume of the perturbed region of Σ under the flow. Differentiating once at $t = 0$ gives the first variation of volume in terms of the the mean curvature

$$\mathcal{V}'(0) = \int_U \mathrm{tr}\bar{k} \cdot v \; d\bar{\mu}, \tag{2.7}$$

where $\bar{k}$ denotes the second fundamental form of Σ, $v = (-\bar{g}(\eta,\eta))^{1/2}$, and $d\bar{\mu}$ is the induced volume form on Σ.

Therefore one finds that when Σ has zero mean curvature, i.e.

$$\mathrm{tr}\,\bar{k} = 0,$$

the first variation of volume is zero. This is true regardless of the variation, i.e. the choice of η above. Under the zero mean curvature assumption, the second variation of volume takes the form

$$\mathcal{V}''(0) = \int_\Sigma -|dv|^2 - (\bar{R}ic(\bar{\nu},\bar{\nu}) + |\bar{k}|^2) \cdot v^2 \, d\bar{\mu}. \tag{2.8}$$

Integration by parts reveals that the second variation can be alternatively written as

$$\mathcal{V}''(0) = \int_\Sigma v \cdot Lv \, d\bar{\mu}, \tag{2.9}$$

where L is the stability operator on Σ defined by

$$Lw = \bar{\Delta} w - \big(|\bar{k}|^2 + \bar{R}ic(\bar{\nu},\bar{\nu})\big) \cdot w.$$

An interesting observation is that (2.8) implies

$$\int_\Sigma w \cdot Lw \, d\bar{\mu} \leq 0,$$

for any nonnegative smooth function $w : \Sigma \to \mathbb{R}$, provided $(M, \bar{g})$ satisfies the so-called timelike convergence condition, $\bar{R}ic(\bar{\nu},\bar{\nu}) \geq 0$. In such a spacetime, this implies that any hypersurface with zero mean curvature will maximize volume under any compactly supported variation as described above, hence the terminology, maximal.

In the case where $(M, \bar{g})$ is a vacuum spacetime and Σ is a maximal hypersurface, one has $\bar{R}ic(\bar{\nu},\bar{\nu}) = 0$ and the Gauss equations for hypersurfaces implies $\bar{s} = |\bar{k}|^2$. One then has

Definition 2.2. *In a vacuum spacetime, the* stability operator *on a maximal hypersurface* Σ *is the differential operator given by*

$$Lw = \bar{\Delta} w - \bar{s} \cdot w.$$

2.3 Maximal Foliations

Let t be a smooth function $t : M \to \mathbb{R}$ free of critical points and having timelike gradient vectorfield $\bar{D}t$. On any hypersurface Σ_s given as the level set $t = s$, the spacetime metric $\bar{g}$ may be decomposed as

$$\bar{g} = -u^2 dt^2 + \bar{h}_s, \tag{2.10}$$

where $u = (-\bar{g}(\bar{D}t, \bar{D}t))^{-1/2}$ is the inverse length of $\bar{D}t$ and $\bar{h}_s$ is the induced metric on Σ_s extended to act trivially on vectors in $T\Sigma^{\perp}$.

Now suppose that M is a vacuum spacetime so that

$$\bar{R}ic \equiv 0$$

Then for a given $\Sigma = \Sigma_s$, the Gauss and Codazzi-Mainardi equations (2.3) and (2.4) for hypersurfaces in a vacuum spacetime become,

$$\bar{s} - |\bar{k}|^2 + (\mathrm{tr}\bar{k})^2 = 0, \tag{2.11}$$

$$\mathrm{div}\,\bar{k} - d(\mathrm{tr}\,\bar{k}) = 0. \tag{2.12}$$

These are called the *constraint equations* for $(\Sigma, \bar{h}, \bar{k})$.

Using the form (2.10) of the metric, one may also derive the following *evolution equations*, relating geometric quantities to the function $u : \Sigma \to \mathbb{R}$ restricted to Σ:

$$\mathcal{L}_{\bar{\nu}}\bar{g} = -2\bar{k}$$
$$u \cdot \mathcal{L}_{\bar{\nu}}\bar{k} = -\nabla du + (\bar{r}ic - 2\,\mathrm{tr}\,\bar{k} \cdot \bar{k} - 2\bar{k}^2) \cdot u,$$

where $\bar{r}ic$ denotes the Ricci tensor determined by $\bar{h}$ on Σ and $k^2 : T\Sigma \otimes \Sigma \to \mathbb{R}$ is given in terms of the orthonormal basis $\{E_i\}$ above as

$$\bar{k}^2 = \sum \bar{k}(E_i,\,\cdot\,)\bar{k}(E_i,\,\cdot\,).$$

Taking the trace of the second equation with respect to $\bar{h}$ and using the constraint equations one obtains

$$\bar{\Delta}u - (2\,(\mathrm{tr}\bar{k})^2 + 2\,|\bar{k}|^2 - \bar{s}) \cdot u = -\mathcal{L}_{\bar{\nu}}(\mathrm{tr}\bar{k}) \cdot u,$$

In the case where each Σ_s is maximal, the above constraint equations become

$$\bar{s} = |\bar{k}|^2, \tag{2.13}$$

$$\operatorname{div} \bar{k} = 0, \tag{2.14}$$

$$\operatorname{tr} \bar{k} = 0. \tag{2.15}$$

So one has $\mathcal{L}_\nu(\mathrm{tr}k) = 0$ and $\bar{s} = |\bar{k}|^2$ on Σ. The above equation for u simplifies to

$$\bar{\Delta}u - |\bar{k}|^2 \cdot u = Lu = 0. \tag{2.16}$$

where L is the stability operator defined above.

This motivates the following definition

Definition 2.3. *Given a maximal hypersurface* $(\Sigma, \bar{h}, \bar{k})$, *the* lapse equation *on* Σ *is given by* $Lu = 0$.

It is necessary that u satisfy the lapse equation in order that the foliation given above be a foliation by maximal hypersurfaces emanating from Σ. u may be thought of as the velocity which the leaves of the foliation spread away from the given maximal leaf Σ.

A reduction procedure (see [11]) can now be performed to transform the above equations to an equivalent system of nonlinear wave equations coupled with the elliptic lapse equation defined on each slice. The maximal gauge therefore simplifies the process solving the Einstein equations to solving an initial value problem where initial data $(\bar{h}, \bar{k})$ is prescribed on some initial slice Σ.

The Cauchy problem involves carrying this procedure through, beginning only with the data $(\Sigma, \bar{h}, \bar{k})$ consisting of a three-manifold Σ, a Riemannian metric $\bar{h}$ defined on Σ and some symmetric two-tensor $\bar{k}$. The goal is to produce a spacetime $(M, \bar{g})$ with metric given by (2.10) and which contains Σ as an embedded spacelike hypersurface for which $\bar{k}$ is its second fundamental form. Of course, to have any hope of success, the above constraint equations will need to be satisfied by the initial data at the outset.

It was a remarkable achievement of Christodoulou and Klainerman [11] to carry this process through for small asymptotically flat initial data, small meaning close to trivial data induced on some spacelike hypersurface in Minkowski space, and asymptotically flat meaning that the data approaches trivial data at large distances. Taking advantage of the simplifications made available by the maximal gauge, they were able to prove the existence of a solution to the Einstein equations which was geodesically complete. An important aspect of the maximal gauge choice is that it allowed them to obtain estimates on the asymptotic behavior of the resulting spacetime solution.

Besides ensuring that the constraint equations are satisfied, one will also want to know that the lapse equation (2.16) is solvable on a given slice Σ. As is shown in [14], on any complete noncompact Riemannian manifold P, the operator L_q given by

$$L_q u = \Delta u - qu,$$

where q is some smooth function on P, admits a positive solution provided the smallest eigenvalue of L_q is nonnegative on P. This is in particular the case when q nonnegative. It follows that given any data $(\Sigma, \bar{h}, \bar{k})$ for the maximal hypersurface gauge in a vacuum spacetime with Σ noncompact and complete, there exists a positive solution to the lapse equation (2.16).

Chapter 3

S^1-Symmetric Spacetimes

3.1 The Quotient Structure

The analysis here is concerned with vacuum spacetimes $(M, \bar{g})$ which admit a free isometric S^1 action whose orbits are spacelike curves. These will be called *S^1-symmetric vacuum spacetimes.* The one parameter family of isometries $\phi_t : \mathbb{R} \times M \to M$ associated with the action corresponds to a spacelike Killing vector field defined at each point p via the relation

$$X_p = \left. \frac{d}{dt} \right|_{t=0} \phi_t(p).$$

A spacelike hypersurface of M is given as data $(\Sigma, \bar{h}, \bar{k})$ with $\bar{h}$ denoting the induced metric and $\bar{k}$ the second fundamental form, as above. The hypersurfaces of interest to us will be those which are preserved by the S^1 action on $(M, \bar{g})$. That is, it is assumed that $\phi_t(\Sigma) = \Sigma$ and

$$\mathcal{L}_X \bar{h} = \mathcal{L}_X \bar{k} = 0.$$

It will also be assumed that Σ is complete and noncompact. Such hypersurfaces will be referred to as S^1-*symmetric hypersurfaces*.

Attention is restricted to the case where the space of orbits corresponding to the S^1 action is a smooth manifold N. Then N inherits a canonical Lorentzian metric g which makes the quotient map

$$\pi_M : (M, \bar{g}) \rightarrow (N, g)$$

a Riemannian submersion (or perhaps "Lorentzian submersion" is more appropriate terminology).

Because $(\Sigma, \bar{h}, \bar{k})$ is S^1-symmetric, π_M restricts to a Riemannian submersion between Σ and its quotient $V = \Sigma/S^1$ which is a submanifold of N:

$$\pi_\Sigma : (\Sigma, \bar{h}, \bar{k}) \rightarrow (V, h, k).$$

The relationship between the second fundamental forms $\bar{k}$ and k will be given below.

The picture to keep in mind is (horizontal arrows are viewed as embeddings).

$$\begin{array}{ccc} (\Sigma, \bar{h}, \bar{k}) & \longrightarrow & (M, \bar{g}) \\ \pi_\Sigma \big\downarrow & & \pi_M \big\downarrow \\ (V, h, k) & \longrightarrow & (N, g) \end{array}$$

The convention will be that barred quantities are defined on the total spaces and the nonbarred quantities are defined on the corresponding quotients. D, Ric and R denote the connection, Ricci curvature, and scalar curvature for N, and ∇, ric and s denote the connection, Ricci curvature, and scalar curvature of V.

Next the geometry of $(M, \bar{g})$ is related to that of (N, g). This is done in terms of two geometric objects. The first is the twist one form $\bar{\omega}$, which reflects the obstruction to integrability of the horizontal distribution of the S^1 bundle $M \to N$. The second is a function f which gives the length of the S^1 fiber above any point in N.

Consider the one-form ξ dual to the Killing vector X on M. The twist one-form associated to X is given by

$$\bar{\omega} = * \xi \wedge d\xi,$$

where $*$ denotes the hodge star operator on $(M, \bar{g})$. By the Frobenius theorem, the condition $\bar{\omega} = 0$ characterizes the property that X is hypersurface orthogonal, or equivalently, that the horizontal distribution of the S^1 bundle M be integrable. As $\bar{\omega}$ is necessarily preserved by the S^1 symmetry and satisfies $\bar{\omega}(X) = 0$, it descends to a one-form ω on N.

It is also useful to consider the function

$$f = |X|_{\bar{g}}$$

giving the length of X. f may and will be assumed to equal the length of the S^1 through any point. This may be arranged through an appropriate parameterization of the diffeomorphisms generating X. f may be viewed either as a function on N or on M.

The vacuum equations $\bar{R}ic \equiv 0$ may be expressed [15] in terms of the above quantities as the following system intrinsic to (N, g):

$$Ric = f^{-1}\nabla df + \frac{1}{2}f^{-4}\left(\omega \otimes \omega - |\omega|^2 g\right), \tag{3.1}$$

$$\Box f = -\frac{1}{2}f^{-3}|\omega|^2, \tag{3.2}$$

$$div(\omega) = 3f^{-1}\langle \omega, df \rangle, \tag{3.3}$$

where metric quantities appearing here are in terms of g on N, $\Box$ denoting the d'Alembertian $tr\nabla^2$.

ξ restricts to a one form η on Σ. Using the hodge star $*$ determined by $\bar{h}$, one can define on Σ the twist scalar

$$\bar{\alpha} = *\,\eta \wedge d\eta.$$

Just as for ω, the condition $\alpha = 0$ corresponds to the integrability of the horizontal distribution for the hypersurface Σ as an S^1 bundle over V. As before, α is taken to represent the corresponding function on the quotient V.

Let $\bar{\nu}$ denote the future directed unit length section of $T\Sigma^{\perp}$. It turns out that

$$\bar{\alpha} = \bar{\omega}(\bar{\nu}) \quad \text{and} \quad \alpha = \omega(\nu),$$

where ν is the future directed unit section of $TV^{\perp}$.

From the above relations, one finds that the scalar curvatures of Σ and V are related as [12]

$$s = \bar{s} + \frac{1}{2} f^{-4}\alpha^2 + 2f^{-1}\Delta f - 2\mathrm{tr}k \cdot d\log f(\nu).$$

In the case where Σ is maximal, this simplifies to

$$s = \bar{s} + \frac{1}{2} f^{-4}\alpha^2 + 2f^{-1}\Delta f. \tag{3.4}$$

Proposition 3.1. *The mean curvatures of the hypersurfaces* $(\Sigma, \bar{h}, \bar{k})$ *and* (V, h, k) *are related by*

$$\mathrm{tr}\bar{k} = \mathrm{tr}k - d\log f(\nu). \tag{3.5}$$

Proof. Let $\pi = \pi_\Sigma$. Begin by choosing an orthonormal frame $\{\bar{E}_0, \bar{E}_1, \bar{E}_2\}$ in $T\Sigma$ such that $\bar{E}_0 = f^{-1}X$ and such that there exist vector fields E_1 and E_2 on V with $E_1(\pi(p)) = \pi_*\bar{E}_1(p)$ and $E_2(\pi(p)) = \pi_*\bar{E}_2(p)$ for each $p \in \Sigma$. In other words, assume $\bar{E}_1$ and $\bar{E}_2$ are basic lifts of E_1 and E_2. Then E_1 and E_2 form an orthonormal basis for V. Note that $\bar{\nu}$ is a basic lift of ν. Extend these vector fields to vector fields on M (or N on the base) which remain basic. Extend $\bar{\nu}$ to be a basic lift of a vector field ν normal to V. Then, for each i,

$$\begin{aligned}\bar{k}(\bar{E}_i, \bar{E}_i) &= -\frac{1}{2}\mathcal{L}_{\bar{\nu}}\bar{g}(\bar{E}_i, \bar{E}_i) \\ &= -\frac{1}{2}\bar{\nu}\bar{g}(\bar{E}_i, \bar{E}_i) + \bar{g}(\mathcal{L}_{\bar{\nu}}\bar{E}_i, \bar{E}_i) \\ &= -\bar{g}(\mathcal{L}_{\bar{E}_i}\bar{\nu}, \bar{E}_i).\end{aligned}$$

Since π_M is a Riemannian submersion, for $i = 1, 2$, one has

$$\bar{g}(\mathcal{L}_{\bar{E}_i}\bar{\nu}, \bar{E}_i) = g(\mathcal{L}_{E_i}\nu, E_i).$$

Using the fact that X is Killing gives

$$\begin{aligned}0 &= \mathcal{L}_X\bar{g}(X, \bar{\nu}) \\ &= X\bar{g}(X, \bar{\nu}) - \bar{g}(\mathcal{L}_X X, \bar{\nu}) - \bar{g}(X, \mathcal{L}_X\bar{\nu}) \\ &= -\bar{g}(X, \mathcal{L}_X\bar{\nu}) \\ &= -\bar{g}(f\bar{E}_0, f\mathcal{L}_{\bar{E}_0}\bar{\nu}) + \bar{g}(f\bar{E}_0, \bar{\nu}(f)\bar{E}_0) \\ &= -f^2\bar{g}(\bar{E}_0, \mathcal{L}_{\bar{E}_0}\bar{\nu}) + f\bar{\nu}(f),\end{aligned}$$

and so

$$\bar{k}(\bar{E}_0, \bar{E}_0) = \bar{g}(\bar{E}_0, \mathcal{L}_{\bar{E}_0}\bar{\nu}) = d\log f(\nu). \tag{3.6}$$

It follows from these observations that

$$\mathrm{tr}\bar{k} = \sum_{i=1}^{3} \bar{k}(\bar{E}_i, \bar{E}_i) = \sum_{i=1}^{2} k(E_i, E_i) + \bar{k}(\bar{E}_0, \bar{E}_0) = \mathrm{tr}k - d\log f(\nu)$$

$\square$

3.2 Some Preliminaries

Several of the results to follow will refer to volume growth on Σ and V. Fix a point p in V, and let $B(r)$ denote the geodesic ball in V of radius r centered at p. Denote by $v(r)$ its volume

$$v(r) = vol_V(B(r))$$

in V, and let

$$\mathbf{v}(r) = vol_\Sigma(\mathcal{B}(r))$$

denote the volume in Σ of the preimage $\mathcal{B}(r) = \pi^{-1}(B(r))$, where π denotes the quotient map $\pi : \Sigma \rightarrow V$. It will be assumed that the boundaries $\partial B(r)$ and $\partial\mathcal{B}(r)$ of these sets are piecewise smooth.

Finally, there are two results which will be useful in the analysis to come. The following proposition was proved by Anderson (Proposition 4.1 in [3])

Proposition 3.2. *Let Σ be a complete noncompact Riemannian manifold with nonnegative scalar curvature. If Σ admits a free isometric S^1 action, then there* *exists a constant c such that the volume of any geodesic ball in $V = \Sigma/S^1$ satisfies*

$$v(r) \leq cr^2, \quad \textit{and} \quad v'(r) \leq cr. \tag{3.7}$$

If $(\Sigma, \bar{h}, \bar{k})$ is a maximal hypersurface in a S^1-symmetric vacuum space-time, then from (2.13) one sees that it satisfies the hypotheses of the above proposition, and geodesic balls in V satisfy the volume growth given above. It will be useful to note that this behavior can be extended to any covering of V since that covering will be a quotient of a covering of Σ.

Several of the results below will appeal to the following lemma motivated by results in [8].

Lemma 3.3. *Let w be a smooth function on V and let $\mathbf{w}$ be a smooth function on Σ. Then*

$$-\int_{B(r)} \Delta w \; d\mu_V \leq [W'(r)v'(r)]^{1/2},$$

where $W(r) := \int_{B(r)} |\nabla w|^2 d\mu_V$, and

$$-\int_{\mathcal{B}(r)} \bar{\Delta}\mathbf{w} \; d\mu_\Sigma \leq [\mathbf{W}'(r)\mathbf{v}'(r)]^{1/2},$$

where $\mathbf{W}(r) := \int_{B(r)} |\bar{\nabla}\mathbf{w}|^2 d\mu_\Sigma$.

Proof. Choose a small $\delta > 0$. Then for $a > 1$ we can find a smooth function η supported on $B(r)$ which satisfies $\eta = 1$ on $B(r - \delta)$ and $|\nabla \eta| \leq \frac{a}{\delta}$. Integration by parts gives

$$
\begin{aligned}
-\int_{B(r)} \eta \cdot \Delta w &= \int_{B(r)\setminus B(r-\delta)} \langle \nabla \eta, \nabla w \rangle \\
&\leq \left[\int_{B(r)\setminus B(r-\delta)} |\nabla \eta|^2 \right]^{1/2} \left[\int_{B(r)\setminus B(r-\delta)} |\nabla w|^2 \right]^{1/2} \\
&\leq \left[\left(\frac{a}{\delta} \right)^2 (v(r) - v(r-\delta)) \right]^{1/2} \left[\int_{B(r)\setminus B(r-\delta)} |\nabla w|^2 \right]^{1/2} \\
&= a \left[\frac{1}{\delta} (v(r) - v(r-\delta)) \right]^{1/2} \left[\frac{1}{\delta} \int_{B(r)\setminus B(r-\delta)} |\nabla w|^2 \right]^{1/2}
\end{aligned}
$$

Taking the limit $\delta \to 0$ we have

$$
-\int_{B(r)} \Delta w \cdot \eta \leq a[v'(r)]^{1/2}[W'(r)]^{1/2}.
$$

Letting $a \to 1$ proves the first inequality.

Imitating the same argument on Σ with $\bar{\eta} := \eta \circ \pi : \Sigma \to \mathbb{R}$ establishes the second inequality. □

3.3 Einstein-Rosen Waves

3.3.1 The Framework

S^1 symmetric spacetimes which admit an additional spacelike Killing field representing a rotational symmetry are known as cylindrical waves. Among these are the Einstein-Rosen waves for which both Killing fields are assumed to commute and are hypersurface orthogonal. A key feature of Einstein-Rosen waves is that they may be determined by solving the wave equation on a flat background Minkowski space. They thus provide one with easy to construct examples of S^1-symmetric spacetimes and can serve as intuition for the general case. Of interest here will be the properties of maximal foliations and the asymptotic nature of spacelike hypersurfaces which, subject to an energy condition, tend to be conical at large distances.

Let $(M, \bar{g})$ be an S^1-symmetric vacuum spacetime for which the twist one-form ω associated with the Killing field X vanishes identically. Then according to (3.1), (3.2) and (3.3) the vacuum equations for $(M, \bar{g})$ are equivalent to the following system of equations on the quotient (N, g):

$$Ric = f^{-1}\nabla df,$$
$$\Box f = 0.$$

Write $f = e^{\psi}$ and consider the conformally rescaled metric $\hat{g} = e^{2\psi}g$. Using standard formulas for the transformation of the Ricci tensor under a conformal change [21], one finds that the vacuum equations become

$$\hat{Ric} = 2d\psi^2$$
$$\Box_{\hat{g}}\,\psi = 0,$$

written in terms of the Ricci tensor $\hat{Ric}$ and d'Alembertian $\Box_{\hat{g}}$ for the rescaled metric $\hat{g}$.

Einstein-Rosen waves are assumed to admit a second hypersurface orthogonal Killing field Y which commutes with X and vanishes precisely along a timelike curve. It therefore represents a rotational symmetry about an axis. Linear combinations of Killing fields are also Killing. Therefore it can be arranged that X and Y are orthogonal. Local coordinates may be found near the timelike axis, here the t-axis, with respect to which the four-dimensional spacetime metric $\bar{g}$ takes the form (c.f. [6]),

$$\bar{g} = e^{2\psi}dz^2 + e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi}d\phi^2 \tag{3.8}$$

The functions ψ and γ depend on ρ and t alone. X is the coordinate vector ∂_z and Y is the coordinate vector ∂_ϕ. These coordinates extend through the connected region where the function $W = |X||Y|$ retains a spacelike gradient. For Einstein-Rosen waves it is generally assumed that this domain encompasses the entire spacetime, diffeomorphic to $S^1 \times \mathbb{R}^3$ (or $\mathbb{R} \times \mathbb{R}^3$).

The conformally rescaled quotient metric reads

$$\hat{g} = e^{2\gamma}(-dt^2 + d\rho^2) + \rho^2 d\phi^2.$$

Evaluating $\hat{Ric}$ on the coordinate vectors reveals

$$\hat{R}ic(\partial_t, \partial_t) = -\ddot{\gamma} + \gamma'' + \rho^{-1}\gamma' = 2\dot{\psi}^2, \tag{3.9}$$

$$\hat{R}ic(\partial_\rho, \partial_\rho) = \ddot{\gamma} - \gamma'' + \rho^{-1}\gamma' = 2\psi'^2, \tag{3.10}$$

$$\hat{R}ic(\partial_t, \partial_\rho) = \rho^{-1}\dot{\gamma} = 2\dot{\psi}\psi', \tag{3.11}$$

where $\dot{}$ and $'$ denote differentiation with respect to t and ρ respectively. The equation $\Box_{\hat{g}}\psi = 0$ takes the surprisingly pleasant form

$$-\ddot{\psi} + \psi'' + \rho^{-1}\psi' = 0.$$

Summing (3.9) and (3.10) gives $\gamma' = \rho(\dot{\psi}^2 + \psi'^2)$. This combined with the remaining equations reduces the vacuum equations for an Einstein-Rosen wave to the system

$$\gamma' = \rho(\dot{\psi}^2 + \psi'^2), \tag{3.12}$$

$$\dot{\gamma} = 2\rho\dot{\psi}\psi', \tag{3.13}$$

$$-\ddot{\psi} + \psi'' + \rho^{-1}\psi' = 0. \tag{3.14}$$

Interestingly, the last equation is nothing more than the axisymmetric wave equation on flat Minkowski space. This makes apparent a straightforward procedure for constructing Einstein-Rosen waves. By prescribing suitable initial data $(\psi, \dot{\psi})$ on a hypersurface given by constant t, one may solve the wave equation (3.14) for ψ on 3-dimensional Minkowski space and then determine the remaining unknown γ using (3.12) and (3.13).

3.3.2 The C Energy

Corresponding to solutions to the wave equation in Minkowski space is the energy of the wave, a conserved quantity obtained by integrating data over a fixed time level set. This can be done for Einstein-Rosen waves. Thorne [20] interpreted this quantity as the total energy for the spacetime and called it the C energy. It turns out that the total C energy can be recorded as a conical defect in the asymptotic geometry of the spacelike hypersurfaces.

In order to produce a wave from initial data, it is most convenient to consider the quotients of each of the level sets Σ_s via the action induced by the Killing field X. This gives surfaces V_s with quotient metric

$$h_s = e^{2(\gamma-\psi)}d\rho^2 + \rho^2 e^{-2\psi}d\phi^2.$$

Consider the Einstein-Rosen wave generated from initial data $\psi(0,\rho)$ and $\dot{\psi}(0,\rho)$ prescribed as functions on V_0. Let $\psi(0,\rho)$ be supported in the disc given by $\{(\rho,\phi) \in V_0 : \rho < \rho_0\}$, and take $\dot{\psi}(0,\rho) \equiv 0$. Consider the wave solution generated from this initial data in Minkowski space. It follows from the theory of the wave that the resulting solution ψ of (3.14) will have support contained in the domain $U = \{(t,\rho,\phi) \mid \rho < t + \rho_0\}$.

The total energy of a wave in Minkowski space is given by

$$E_c := \int_0^\infty \rho(\dot{\psi}(s,\rho)^2 + \psi'(s,\rho)^2)d\rho.$$

This is a conserved quantity on the hypersurfaces V_s, meaning that it does not depend on s. Thorne [20] refers to this as the total C energy for the Einstein-Rosen wave. In light of equation (3.12), one finds that γ takes the constant value $\gamma = E_c$ outside of U. Therefore on each V_s, outside of U, the metric reads

$$h_s = e^{2E_c}d\rho^2 + \rho^2 d\phi^2.$$

This is exactly the metric induced on a flat cone in 3-dimensional Euclidean space given as the solution to the equation $z = (e^{2E_c} - 1)^{1/2}r$ in cylindrical coordinates (z, r, θ). This relationship between the energy and the conical behavior of the hypersurfaces will be of interest later.

3.3.3 Maximal Hypersurfaces

As the analysis to come is concerned with the maximal hypersurface gauge, it will be interesting to explore the properties of hypersurface foliations in Einstein-Rosen waves. The most natural spacelike foliation to consider in this context is given by the level sets $\Sigma_s \subset M$ obtained by setting $t = s$. To match the analysis for the general S^1-symmetric setting, one would like this foliation to be maximal. It turns out that while one can arrange that a single hypersurface be maximal, it is impossible to preserve this maximality on neighboring hypersurfaces of this form unless the spacetime is trivial or has singularities.

Recall (2.2), that the second fundamental form $\bar{k}$ on a given $\Sigma = \Sigma_s$ is given in terms of the unit normal vector field $\bar{\nu} = e^{\psi-\gamma}\partial_t$ by

$$\bar{k} = -\frac{1}{2}\mathcal{L}_{\bar{\nu}}\bar{h},$$

where $\bar{h}$ is the metric on Σ given by

$$\bar{h} = e^{2\psi}dz^2 + e^{2(\gamma-\psi)}d\rho^2 + e^{-2\psi}\rho^2 d\phi^2.$$

An orthonormal basis for $\bar{h}$ is $\{E_1, E_2, E_3\} = \{e^{-\psi}\partial_z, e^{\psi-\gamma}\partial_\rho, \rho^{-1}e^{\psi}\partial_\phi\}$. Direct computation gives

$$\begin{aligned}
\bar{k}(E_1, E_1) &= -\frac{1}{2}e^{-2\psi}\mathcal{L}_{\bar{\nu}}\bar{h}(\partial_z, \partial_z) = -\frac{1}{2}e^{-\psi-\gamma}\partial_t\bar{h}(\partial_z, \partial_z) = -\dot{\psi}e^{\psi-\gamma} \\
\bar{k}(E_2, E_2) &= -\frac{1}{2}e^{2(\psi-\gamma)}\mathcal{L}_{\bar{\nu}}\bar{h}(\partial_\rho, \partial_\rho) = -\frac{1}{2}e^{3(\psi-\gamma)}\partial_t\bar{h}(\partial_\rho, \partial_\rho) = (\dot{\psi} - \dot{\gamma})e^{\psi-\gamma} \\
\bar{k}(E_3, E_3) &= -\frac{1}{2}\rho^{-2}e^{2\psi}\mathcal{L}_{\bar{\nu}}\bar{h}(\partial_\phi, \partial_\phi) = -\frac{1}{2}\rho^{-2}e^{3\psi-\gamma}\partial_t\bar{h}(\partial_\phi, \partial_\phi) = \dot{\psi}e^{\psi-\gamma}
\end{aligned}$$

From these relations one can compute the trace and metric norm of $\bar{k}$ with respect to $\bar{h}$ to find

$$\mathrm{tr}\bar{k} = (\dot{\psi} - \dot{\gamma})e^{\psi-\gamma},$$
$$|\bar{k}|^2 = \left[2\dot{\psi}^2 + (\dot{\psi} - \dot{\gamma})^2\right] e^{2(\psi-\gamma)}.$$

Notice that the hypersurface Σ is maximal precisely when $\dot{\psi} = \dot{\gamma}$ on Σ. According to equation (3.13) this will necessarily be the case where $\dot{\psi}$ is zero. Where $\dot{\psi}$ is not zero, dividing (3.13) through by $\dot{\psi}$ shows that ψ must be logarithmic as a function of ρ. Consider how this fact relates to the C energy. One can not insist on bounded energy on a maximal hypersurface and at the same time allow $\dot{\psi}$ to remain nonzero over too large a region of that hypersurface since logarithmic growth in ψ is not compatible with a bounded C energy.

Conversely, a bound on the C energy directly correlates to conical asymptotics. This suggests that the quotients of maximal hypersurfaces in a *general* S^1-symmetric spacetime might necessarily be asymptotically conical, perhaps if there exists a bound on quantities analogous to the total C energy. If this were the case, it would eliminate the necessity of imposing such an asymptotic condition on Cauchy data. The analogous quantity is proposed in the section on Energy below.

It is also worth pointing out that while one can arrange that a given hypersurface Σ_s be maximal through the appropriate assignment of data on that hypersurface, the evolution equations (3.12), (3.13) and (3.14) will not permit maximality to be preserved from one hypersurface to the next, at least not globally on the hypersurface in a smooth way.

Even though the focus of this work is on foliations of the four-dimensional spacetime, as a side note, it is also interesting to consider the second fundamental forms of the quotient hypersurfaces V given by

$$\mathrm{tr}k = (2\dot{\psi} - \dot{\gamma})e^{\psi-\gamma},$$
$$|k|^2 = \left[\dot{\psi}^2 + (\dot{\psi} - \dot{\gamma})^2\right] e^{2(\psi-\gamma)}.$$

Maximality of V corresponds to $2\dot{\psi} = \dot{\gamma}$, but as in the nonquotient case, the condition $\dot{\psi} = 0$ will do, and the discussion above goes through in the quotient setting verbatim.

3.4 Main Results

In what follows, it is assumed that the length of the Killing vector field X generated by the S^1-symmetry is bounded above

$$0 < f < f_0$$

In the nonsymmetric setting, for asymptotically flat data, it is typically assumed that the lapse function u has the asymptotic behavior $u \to 1$. The following theorem reveals that in the S^1-symmetric setting such a condition has strong consequences on the data.

Theorem 3.4. *Suppose that maximal S^1-symmetric Cauchy data $(\Sigma, \bar{h}, \bar{k})$ ad-*

mits a bounded solution u of the lapse equation $Lu = 0$. Then the data is totally geodesic, i.e. $\bar{k} \equiv 0$, and u is constant.

Proof. As in the previous section, let $B(r)$ denote the geodesic ball of radius r with volume $v(r)$. Then the condition

$$\int^{\infty} \frac{1}{\mathbf{v}'(r)}\, dr = \infty \tag{3.15}$$

implies that any subharmonic function on Σ which is bounded above is necessarily constant on Σ. The proof of this fact follows the arguments in [8] and is as follows. Let u be any such function bounded above by u_0, so that $w := u_0 - u$ is a positive superharmonic function. In particular,

$$w^{-1}\bar{\Delta} w = \bar{\Delta} \log w + |d \log w|^2 \leq 0,$$

so we have $|d \log w|^2 \leq -\bar{\Delta} \log w$. Integrating both sides of this inequality over $\mathcal{B}(r)$ and applying Lemma 3.3 gives

$$W(r) \leq [W'(r)\mathbf{v}'(r)]^{1/2},$$

where $W(r) := \int_{\mathcal{B}(r)} |d \log w|^2 \, d\bar{\mu}_{\Sigma}$.

If $W(r) = 0$ for all r, then the statement is proved, so assume there exists r_0 so that $W(r) > 0$ whenever $r > r_0$. Then on the interval (r_0, ∞) the above inequality may be rewritten

$$\frac{W'}{W^2} \geq \frac{1}{\mathbf{v}'}.$$

Integrating each side over the interval (r_0, r) gives

$$\frac{1}{W(r_0)} \geq \int_{r_0}^{r} \frac{W'}{W^2}\, dr \geq \int_{r_0}^{r} \frac{1}{\mathbf{v}'}\, dr.$$

This is incompatible with (3.15), so given (3.15) we must have $W \equiv 0$, i.e. u is constant on Σ.

The Lapse equation reads

$$\bar{\Delta} u = |\bar{k}|^2 u.$$

From the Gauss equation for spacelike hypersurfaces in a vacuum spacetime we have $\bar{s} = |\bar{k}|^2 \geq 0$, so u is subharmonic. Because f is assumed to be bounded, $\mathbf{v}'(r) \leq cr$ for some c and for all r, which establishes condition (3.15). If u is bounded it must be constant. Returning to the lapse equation, one sees that this implies $|\bar{k}|^2 \equiv 0$, i.e. Σ is totally geodesic. □

The corollary below demonstrates that imposing the maximal hypersurface gauge with a bounded lapse can only lead to a static spacetime development. A spacetime is *static* provided it admits a hypersurface orthogonal timelike Killing field. A static spacetime splits as a trivial product $\mathbb{R} \times \Sigma$ with its metric expressible as

$$\bar{g} = -dt^2 + pr^*\bar{h},$$

where $\bar{h}$ is a fixed metric on the Riemannian manifold Σ and $pr : \mathbb{R} \times \Sigma \rightarrow \Sigma$ is the trivial projection.

Corollary 3.5. *If* $(M, \bar{g})$ *is foliated by maximal* S^1*-symmetric hypersurfaces* $(\Sigma, \bar{h}, \bar{k})$*, each admitting a bounded solution to the lapse equation, then* $(M, \bar{g})$ *is static.*

Proof. Let $\bar{\nu}$ denote a unit vector field normal to each maximal hypersurface. Then given any vector fields W, Y tangent to a given hypersurface Σ, we have $\mathcal{L}_{\bar{\nu}}\bar{g}(W, Y) = 2k(W, Y) = 0$ by Theorem 3.4. Furthermore, for any vector field $Z \in TM$ we have

$$\begin{aligned}
\mathcal{L}_{\bar{\nu}}\bar{g}(Z, \bar{\nu}) &= \bar{D}_{\bar{\nu}}\bar{g}(Z, \bar{\nu}) + \bar{g}(\bar{D}_Z\bar{\nu}, \bar{\nu}) \\
&= \bar{g}(\bar{D}_Z\bar{\nu}, \bar{\nu}) \\
&= Z\bar{g}(\bar{\nu}, \bar{\nu}) - \bar{g}(\bar{\nu}, \bar{D}_Z\bar{\nu}) \\
&= -\bar{g}(\bar{\nu}, \bar{D}_Z\bar{\nu})
\end{aligned}$$

Equality between the second and fourth lines implies that $\mathcal{L}_{\bar{\nu}}\bar{g}(Z, \bar{\nu}) = 0$. Therefore the normal vector field $\bar{\nu}$ is a Killing field and $(M, \bar{g})$ is static. □

Since any geodesically complete static solution to the vacuum Einstein equations must be isometric to the flat Minkowski space, or a quotient by a discrete group of isometries [1], Corollary 3.5 implies that one must allow an unbounded lapse on order to obtain any nontrivial complete S^1-symmetric spacetime in the maximal hypersurface gauge.

It is natural to ask then what behavior of the lapse is acceptable in the presence of S^1 symmetry. The next lemma gives some mild control on solutions to the lapse equation and the second statement is a bound which will be used later.

Lemma 3.6. *Suppose that u is a solution to the lapse equation for maximal S^1-symmetric Cauchy data $(\Sigma, \bar{h}, \bar{k})$. Then*

$$\int_\Sigma |\bar{\nabla} \log u|^2 \, d\bar{\mu}_\Sigma < \infty, \tag{3.16}$$

and

$$\int_\Sigma |\bar{\nabla} \log f|^2 + \bar{s} + f^{-4}\alpha^2 \, d\bar{\mu}_\Sigma < \infty. \tag{3.17}$$

Proof. From the constraint equation (2.13), the lapse equation for a maximal hypersurface may be written

$$u^{-1}\bar{\Delta} u = \bar{s}$$

Relating $\bar{s}$ to s by (3.4) and lifting S^1 invariant quantities to Σ, one finds

$$2s = \bar{\Delta} \log u + |\bar{\nabla} \log u|^2 + 4\bar{\Delta} \log f + 4|\bar{\nabla} \log f|^2 + \bar{s} + f^{-4}\alpha^2.$$

Put

$$G(r) := \int_{\mathcal{B}(r)} |\bar{\nabla} \log u|^2 + 4|\bar{\nabla} \log f|^2 + \bar{s} + f^{-4}\alpha^2 \, d\bar{\mu}_\Sigma,$$

and integrate both sides of the above inequality over $\mathcal{B}(r)$. Lemma 3.3 gives

$$\int_{\mathcal{B}(r)} 2s \, d\bar{\mu}_\Sigma \geq G - 2[G'\mathbf{v}']^{1/2}$$

Since the length of the S^1 fibers satisfies $f < f_0$, volumes are related as

$$\mathbf{v}'(r) = \int_{\partial B(r)} f \, d\mu_{\partial B} \leq f_0 \cdot v'(r).$$

Therefore the above inequality may be expressed on V by

$$f_0 \cdot \int_{B(r)} 2s \, d\mu_V \geq G - 2[f_0 G' v']^{1/2}$$

Since the the scalar curvature s is twice the Gauss curvature, the Gauss-Bonnet theorem gives

$$\int_{B(r)} s \, d\mu_V = 4\pi\chi(B(r)) - 2\kappa(r),$$

where $\kappa(r)$ denotes the total geodesic curvature of the piecewise smooth boundary $\partial B(r)$. This is the integral of the geodesic curvature of the curves which define $\partial B(r)$ plus the sum of exterior angles [7], and $\chi(B(r))$ is the Euler characteristic of $B(r)$. If $l(t)$ denotes the length of $\partial B(r)$ then

$$v''(r) = l'(r) \leq \kappa(r).$$

This is the one dimensional analogue of the first variation of volume (2.7), where the hypersurfaces (with volume $l(r) = v'(r)$) are curves representing the boundary $\partial B(r)$, so it is the first variation of length in this case. Geodesic curvature takes the place of mean curvature (c.f. [17]). The inequality would become equality if all exterior angles were nonnegative.

Note that V is orientable since the orientation of Σ along with the nonvanishing vector field X induces an orientation on V. So one has $\chi(B(r)) \leq 1$. It follows that

$$f_0(8\pi - 4v'') \geq G - 2\left[f_0 G' v'\right]^{1/2}.$$

If $G(r) \leq 8\pi$ for all r, then the statement is proved, so we assume $G(r) > 8\pi f_0$ for large r (note that $G(r)$ is an increasing function). Then setting $\tilde{G} = G - 8\pi f_0$, one has

$$\frac{2(f_0\tilde{G}')^{1/2}}{\tilde{G}} \geq \frac{1}{(v')^{1/2}} + \frac{4f_0 v''}{(v')^{1/2}\tilde{G}}.$$

The trick is to integrate both sides from r_0 to r. Applying the Cauchy-Schwarz inequality to the term of the left gives

$$\begin{aligned}
\int_{r_0}^{r} \frac{2(f_0\tilde{G}'(t))^{1/2}}{\tilde{G}(t)}\, dt &\leq \left(\int_{r_0}^{r} \frac{4f_0\tilde{G}'(t)}{\tilde{G}(t)^2}\, dt\right)^{1/2} \left(\int_{r_0}^{r} dt\right)^{1/2} \\
&= 2\sqrt{f_0}\left(\frac{1}{\tilde{G}(r_0)} - \frac{1}{\tilde{G}(r)}\right)^{1/2} (r - r_0)^{1/2} \\
&\leq 2\sqrt{f_0}\left(\frac{1}{\tilde{G}(r_0)}\right)^{1/2} r^{1/2}
\end{aligned}$$

Integrating the other terms and using the bound $v'(t) \leq ct$ from Proposition 3.2 gives

$$\int_{r_0}^{r} \frac{1}{v'(t)^{1/2}}\, dt \geq \frac{2}{c^{1/2}}(r^{1/2} - r_0^{1/2})$$

and integration by parts on the last term yields

$$\begin{aligned}\int_{r_0}^{r} \frac{4 f_0 v''}{(v')^{1/2}\tilde{G}}\, dt &\geq 8 f_0 \left(\frac{v'(r)^{1/2}}{\tilde{G}(r)} - \frac{v'(r_0)^{1/2}}{\tilde{G}(r_0)} \right) + 8 \int_{r_0}^{r} \frac{v'(t)^{1/2}}{\tilde{G}(t)^2} \tilde{G}\,'(t)\, dt \\ &\geq -8 f_0 \frac{v'(r_0)^{1/2}}{\tilde{G}(r_0)}\end{aligned}$$

Putting these together shows

$$2\sqrt{f_0}\left(\frac{1}{\tilde{G}(r_0)}\right)^{1/2} r^{1/2} \geq \frac{2}{c^{1/2}}(r^{1/2} - r_0^{1/2}) - 8 f_0 \frac{v'(r_0)^{1/2}}{\tilde{G}(r_0)}$$

Diving through by $r^{1/2}$ and letting $r \to \infty$, one finds

$$[\tilde{G}(r_0)]^{-1/2} \geq (c f_0)^{-1/2},$$

This is true for all r_0 sufficiently large. This implies that G is bounded and completes the proof. □

When dealing with Cauchy data, one may consider the possibility that the Cauchy surface consists of several ends. An end of Σ is determined by choosing a compact subset of Σ and is defined to be an unbounded component of $\Sigma \backslash \Omega$. The existence of multiple ends and the properties of the ends are a concern when dealing with Cauchy data. The next theorem illustrates that nontrivial data can consist of only of a single end. Furthermore, the geometry of the quotient (V, h) is such that it is conformal to the Euclidean plane, and because $\mathbb{R}^2$ is contractible, the S^1 bundle Σ must then be the topologically trivial product $S^1 \times V$.

Theorem 3.7. *Given maximal S^1-symmetric Cauchy data $(\Sigma, \bar{h}, \bar{k})$, Σ is either the trivial product $S^1 \times S^1 \times \mathbb{R}$ or topologically $S^1 \times V$ with (V, h) conformal to the Euclidean plane.*

Proof. First consider the case where V is simply connected. According to the uniformization theory for surfaces (see [13]), (V, h) is conformal either to the Euclidean plane, the hyperbolic plane, or the sphere. Proposition 3.2 states that volumes $v(r)$ of geodesic balls in V satisfy $v'(r) \leq cr$ for some constant c. In particular we have

$$\int^{\infty} \frac{1}{v'(r)} dr = \infty,$$

which, as shown in [8], or by adapting the discussion in the proof of Theorem 3.4, implies that the only subharmonic functions on V which are bounded above must be constant. The sphere is ruled out since V is noncompact and the hyperbolic plane admits bounded subharmonic functions. Therefore V must be conformal to the Euclidean plane.

Now, if V is not simply connected then V is topologically equal to $S^1 \times \mathbb{R}$ since its only admissible topologies are $\mathbb{R} \times S^1$ or $S^1 \times S^1$ and it is assumed that V is noncompact.

Recall relation (3.4):

$$\bar{s} + 2|d \log f|^2 + \frac{1}{2} f^{-4} \alpha^2 = s - 2\Delta \log f.$$

Put

$$F(r) := \int_{B(r)} \bar{s} + |d \log f|^2 + \frac{1}{2} f^{-4} \alpha^2 \; d\mu_V,$$

and integrate both sides of the above inequality over a geodesic ball $B(r)$ of radius r. Note that V is orientable since the orientation of Σ along with the nonvanishing vector field X induces an orientation on V. Therefore, choosing r large enough guarantees that $\chi(B(r)) \leq 0$. Then Lemma 3.3 implies

$$F(r) \leq \int_{B(r)} s \; d\mu_V + [F'(r) v'(r)]^{1/2} \tag{3.18}$$

$$\leq -2v''(r) + [F'(r) v'(r)]^{1/2}. \tag{3.19}$$

It follows from Proposition 3.2 that the universal cover $\tilde{V}$ of V has at most quadratic volume growth. Since V is a $\mathbb{Z}$ quotient of $\tilde{V}$, volume growth in V must in fact satisfy $v(r) \leq cr$, for some constant c [2]. Using this fact and integrating the above inequality over an interval $[t_0, t]$, t_0 sufficiently large, gives

$$\int_{t_0}^{t} F(r)\,dr \le 2v'(t_0) - 2v'(t) + \int_{t_0}^{t} [F'(r)v'(r)]^{1/2}\,dr \tag{3.20}$$

$$\le 2v'(t_0) + \left[\int_{t_0}^{t} F'(r)\,dr\right]^{1/2} \left[\int_{t_0}^{t} v'(r)\,dr\right]^{1/2} \tag{3.21}$$

$$\le 2v'(t_0) + [F(t) - F(t_0)]^{1/2}\,[cr]^{1/2} \tag{3.22}$$

Lemma 3.6 tells us that F is bounded, so the right hand side grows less than linearly. Since F is a nonnegative nondecreasing function, $F(t_0)$ must be zero, otherwise the left hand side would grow at least linearly. This is true for all t_0 sufficiently large, and thus for all t, so it must be that $F \equiv 0$. This implies that f is constant and $\alpha = s = 0$, so the surface V is flat and the bundle Σ is trivial. □

3.5 Energy

Recall the total C energy E_c for an Einstein-Rosen wave determined by initial data $(\dot{\psi}, \psi)$ prescribed on the $t = 0$ hypersurface $V = V_0$:

$$E_c = \gamma(\infty) = \int_0^{\infty} (\dot{\psi}^2 + \psi'^{\,2})\rho\,d\rho.$$

As discussed above, when this quantity is finite it corresponds to the cone angle inherent in the asymptotic geometry of V. It would be interesting if there were an analogous quantity in the general S^1-symmetric setting. This would give one information regarding the asymptotic behavior of hypersurfaces and would indicate that such asymptotics cannot be prescribed freely.

The integrand was derived in terms of the Ricci tensor $\hat{R}ic_c$ and the scalar curvature $\hat{R}_c$ for the conformally rescaled three-dimensional spacetime metric $\hat{g}_c = e^{2\psi} g_c$ by

$$e^{2\gamma} \left(2\hat{R}ic_c(\hat{\nu}, \hat{\nu}) + \hat{R}_c \right) = \hat{R}ic_c(\partial_t, \partial_t) + \hat{R}ic_c(\partial_\rho, \partial_\rho) = 2\dot{\psi}^2 + 2\psi'^{\,2},$$

where $\hat{\nu}_c = e^{-\gamma}\partial_t$ denotes the unit vector field normal to the $t =$ constant hypersurfaces with respect to $\hat{g}_c$. Expressing E_c as an integral over V with respect to the volume form $d\mu_c = e^{\gamma - 2\psi} \rho \; d\rho \wedge d\phi$ given by the hypersurface metric h_c gives

$$E_c = \frac{1}{4\pi} \int_{V_0} e^{\gamma + 2\psi} \left(2\hat{R}ic_c(\hat{\nu}, \hat{\nu}) + \hat{R}_c \right) \, d\mu_c.$$

The factor $1/4\pi$ arises in part because from the factor of 2 which appears on the right hand side of the above equation and also from integration in the coordinate ϕ.

One would like to find an analogous quantity for the general S^1-symmetric setting. However, there is no γ in the general setting. To get around this, consider the quantity $\mathbf{e}_c$ given as

$$\mathbf{e}_c = e^{2\psi} \left(2\hat{R}ic_c(\hat{\nu}, \hat{\nu}) + \hat{R}_c \right).$$

Integrating this over the ball $B(r) = \{\rho < r\}$ in V gives

$$\begin{aligned}\frac{1}{4\pi}\int_{B(r)} \mathbf{e}_c \, d\mu_V &= \int_0^r e^{-\gamma}(\dot{\psi}^2 + \psi'^2)\rho \, d\rho \\ &= \int_0^r \gamma' e^{-\gamma} \, d\rho \\ &= 1 - e^{-\gamma(r)}\end{aligned}$$

Thus

$$\gamma(r) = -\log\left(1 - \frac{1}{4\pi}\int_{B(r)} \mathbf{e}_c \, d\mu_V\right).$$

This illustrates that obtaining an upper bound on γ, and thus the asymptotic cone angle of hypersurfaces, is equivalent to bounding the integral which appears on the right by a number less than 4π.

The analogous quantity in the general S^1-symmetric setting is obtained by rescaling the quotient spacetime metric to produce $\hat{g} = f^2 g$ on N. The unit vector field normal to the surface V is given by $\hat{\nu} = f^{-1}\nu$. Using standard formulas for the change in curvature under conformal rescalings one finds

$$2\hat{R}ic(\hat{\nu}, \hat{\nu}) + \hat{R} = f^{-2}\mathbf{e},$$

where $\mathbf{e}$ is given on V by

$$\mathbf{e} = 2|d\log f|_V^2 + 2d\log f(\nu)^2 + \frac{1}{2}f^{-4}\left(|\omega|_V^2 + \omega(\nu)^2\right).$$

Thus, the quantity analogous to γ for general S^1-symmetric spacetimes is given by integrating $\mathbf{e}$ over the geodesic ball $B(r)$ in V:

$$\Gamma(r) := \int_{B(r)} \mathbf{e}\, d\mu_V$$

One may hope that bounding this quantity above by a number less than 4π would imply conical asymptotics for V. While this has not been established, the Theorem below shows that, in the case where $d \log f(\nu) = 0$, 4π is an upper bound for Γ. It also shows that when $\mathbf{e}$ is nonzero, volumes asymptotically grow at a rate no less than in Euclidean space.

Theorem 3.8. *Suppose* $(\Sigma, \bar{h}, \bar{k})$ *is maximal* S^1*-symmetric data and* $\mathbf{e}$ *is not identically zero on* V*. Then*

$$\lim_{r\to\infty} v'(r)/r < 2\pi,$$

and if $d \log f(\nu) \equiv 0$ *on* V*, then*

$$\Gamma \leq 4\pi. \tag{3.23}$$

Proof. Using the formulation of the vacuum equations given by equations (3.1) and (3.2) on N, one has

$$\begin{aligned} 2Ric(\nu,\nu) &= 2f^{-1}Ddf(\nu,\nu) + f^{-4}|\omega|_V^2 \\ &= 2f^{-1}\Delta f + 2d\log f(\nu)^2 + 2f^{-4}|\omega|_V^2 - f^{-4}\omega(\nu)^2, \end{aligned}$$

and taking the trace of (3.1) gives

$$R = -\frac{3}{2}f^{-4}|\omega|_N^2 = -\frac{3}{2}f^{-4}|\omega|_V^2 + \frac{3}{2}f^{-4}\omega(\nu)^2.$$

Summing and using (3.2) gives

$$R + 2Ric(\nu,\nu) = 2f^{-1}\Delta f + 2d\log f(\nu)^2 + \frac{1}{2}f^{-4}\left(|\omega|_V^2 + \omega(\nu)^2\right)$$

From the Gauss equation and (3.5) we have

$$\begin{aligned} R + 2Ric(\nu,\nu) &= s - |k|^2 + (trk)^2 \\ &\le s + (trk)^2 \\ &= s + d\log f(\nu)^2 \end{aligned}$$

Combining the above gives

$$s \ge 2f^{-1}\Delta f + d\log f(\nu)^2 + \frac{1}{2}f^{-4}\left(|\omega|_V^2 + \omega(\nu)^2\right). \qquad (3.24)$$

Let $B(r)$ denote a geodesic ball of radius r in V. Then emulating the proof of Lemma 3.6 with

$$G(r) := \int_{B(r)} 2|\nabla \log f|^2 + d\log f(\nu)^2 + \frac{1}{2} f^{-4} \left(|\omega|_V^2 + \omega(\nu)^2\right)\; d\mu_V,$$

shows that G is bounded. However, more can be said from the inequality

$$G(r) \leq 4\pi - 2v''(r) + [G'(r)v'(r)]^{1/2} \tag{3.25}$$

If $G(r) < 4\pi$ for all r, then the proof is complete, so assume otherwise. Then because G is nondecreasing, there exists an r_0 so that $G(r) \geq 4\pi$ for all $r > r_0$. Define $\tilde{G}(r) = G(r) - 4\pi$. Now, observe that the integral

$$\int_{r_0}^{r} \left[\tilde{G}'(t)v'(t)\right]^{1/2}\; dt$$

grows "less than linearly" as a function of r. This follows from (3.2) which gives a constant c such that $v'(r) \leq cr$ and the Cauchy-Schwarz inequality which gives

$$\begin{aligned}
\int_{r_0}^{r} [\tilde{G}'(t)v'(t)]^{1/2}\, dt &\leq \left(\tilde{G}(r) - \tilde{G}(r_0)\right)^{1/2} \left(\frac{c}{2}r^2 - \frac{c}{2}r_0^2\right)^{1/2} \\
&\leq \left(\tilde{G}(r) - \tilde{G}(r_0)\right)^{1/2} (c/\sqrt{2})r
\end{aligned}$$

$\tilde{G}$ is nondecreasing and so converges to its upper bound. Therefore, given any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$ sufficiently large so that $(\tilde{G}(r) - \tilde{G}(r_0))^{1/2} < \epsilon$ on the interval $[r_0, \infty)$. This implies that

$$\int_{r_0(\epsilon)}^{r} [\tilde{G}'(t)v'(t)]^{1/2}\, dt \leq \epsilon^{1/2}(c/\sqrt{2})r.$$

Return to (3.25) and integrate both sides from r_0 to r,

$$\int_{r_0}^{r} \tilde{G}(t)\, dt \leq 2v'(r_0) + \epsilon^{1/2}(c/\sqrt{2})r. \tag{3.26}$$

$\tilde{G}$ is nondecreasing, so the left hand side of the above inequality is no less than $\tilde{G}(r_0)r$ for all $r > r_0$. In order to ensure that the above inequality holds, one must therefore have $\tilde{G}(r_0(\epsilon)) \leq \epsilon^{1/2}(c/\sqrt{2})$. Take a sequence of $\epsilon \to 0$. Then $\tilde{G}(r_0(\epsilon)) \to 0$, but $\tilde{G}$ is nondecreasing, so this implies $\tilde{G} \equiv 0$ and $G \leq 4\pi$.

Returning to (3.25), one has

$$v''(r) \leq 2\pi - \frac{1}{2}G(r) + \frac{1}{2}[G'(r)v'(r)]^{1/2}$$

Integrating both sides of this inequality and appealing to the sublinearity of $\int_{r_0}^{r} [\tilde{G}'(t)v'(t)]^{1/2}\, dt$ establishes the statement involving v'. □

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Appendix A

A Solution to the Lapse Equation

Let $(\Sigma, \bar{h}, \bar{k})$ be S^1-symmetric data which is noncompact and complete. In this section it is shown that there exists an S^1-invariant solution u to the lapse equation

$$Lu = \bar{\Delta}u - \bar{s}u = 0. \tag{A.1}$$

If w is a function on Σ which is S^1 invariant, i.e. $dw(X) = 0$, then w may be considered as a function on the quotient manifold $V = \Sigma/S^1$. In this case, $Lw = 0$ may be expressed on V as

$$Lw = \Delta w + h(\nabla \log f, \nabla w) - \bar{s} \cdot w = 0, \tag{A.2}$$

where $\bar{s}$ still denotes the scalar curvature of Σ, but is viewed as a function on V.

Let $B(r)$ be a geodesic ball of radius r centered at some point $p \in V$. From standard theory for elliptic differential equations (c.f. [16], Theorem 8.14), there exists a smooth solution to the Dirichlet problem

$$\begin{aligned} Lw &= 0 \quad \text{on } B(r) \\ w &= 1 \quad \text{on } \partial B(r). \end{aligned}$$

It follows from the strong maximum principle ([16] Theorem 3.5) that $w > 0$.

For any $r = r_k \in \mathbb{N}\backslash\{0\}$, let $\tilde{u}_k$ denote the solution to the Dirichlet problem. Set $u_k := \tilde{u}_k/\tilde{u}_k(p)$ so that $u_k(p) = 1$ for all k. Now, for a fixed r, the

Harnack inequality ([16], Theorem 8.20) gives us a constant C so that for all k sufficiently large, each function u_k restricted to $B(r)$ satisfies

$$u_k \leq C \quad \text{on } B(r).$$

Then by elliptic regularity ([16] Theorem 6.2), one can control the derivatives of the u_k and use the Arzela-Ascoli theorem to find a subsequence $\{u_k\}$ which converges to a smooth function u_r satisfying $Lu_r = 0$ on $B(r)$. This can be done for a every r in a sequence r_k tending to ∞. By a diagonalization argument, one obtains a limiting solution u on V which lifts to an S^1-invariant solution on Σ.