ON THE TIME-BANDWIDTH CONCENTRATION OF SIGNAL FUNCTIONS FORMING GIVEN GEOMETRIC VECTOR CONFIGURATIONS

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On The Time-Bandwidth Concentration of Signal Functions Forming Given Geometric Vector Configurations

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Summary—Landau, Pollak, and Slepian, [4]-[6] have shown that the prolate spheroidal wave functions play an important role in determining the approximate dimensionality of a space of functions whose energies are concentrated in a given time bandwidth \( WT \). They have also shown the extent to which this space may be assumed 1 \( WT \) dimensional. The function space which they consider is actually infinite dimensional and a subset of \( \mathcal{L}_2 \), but it is not a linear subspace of \( \mathcal{L}_2 \). In general does it necessarily contain any linear subspace of dimensionality 2 \( WT \).

However, in the problem of the discrete M-nary channel with additive Gaussian noise and perhaps other types of noise, one is mainly concerned with given \( n \)-dimensional linear subspaces of \( \mathcal{L}_2 \) and given geometric configurations of vectors in these subspaces. Thus to be conveniently applied to this problem, the results of Landau, Pollak and Slepian should be reformulated in terms of arbitrary given finite dimensional linear subspaces of \( \mathcal{L}_2 \) with given geometric configurations therein. This paper undertakes such a reformulation for some important special cases.

In particular, for the cases of orthogonal, biorthogonal and simplex configurations, it is shown that one can orient the configuration such that the time-bandwidth concentration of the least concentrated vector in the configuration is maximized. The maxi-min criterion is chosen because, as is also shown, the average concentration for these three configurations is always independent of orientation.

I. INTRODUCTORY REMARKS

Much of modern detection theory was formulated under the impetus of wartime and early post-war researches into radar. The problems involved echoes from large, relatively slow, and widely separated targets. In the context of that problem, everyone was well aware that when the theory required the limits of a convolution integral over the time variable to be extended to plus and minus infinity, this really meant: "far enough to include virtually all the return associated with a single echo—but not so far as to include substantial parts of the returns from other echoes". Usually, this statement was practically meaningful. The separation between echoes was normally many times the nominal duration of a single echo. Similar integrals in the frequency domain could have their limits extended over an "effectively infinite" bandwidth without picking up interference from other radars or man-made sources of electromagnetic radiation because the narrow antenna beamwidth, generally high pointing angles and often the geographical remoteness of the actual systems considered gave a high degree of spatial isolation that left the usable spectrum relatively uncrowded. Unfortunately, these convenient circumstances generally have never existed in the field of communications problems. With today's dense, high velocity "threat clouds" and the associated ambiguity and discrimination problems, spatial isolation can no longer so blithely be presumed in the context of many radar problems.

In theory, the signal functions of the communications problem and the echo functions of the multitarget radar problem can be made identically zero outside some arbitrarily specified time interval. An integration of a single-signal function over limits just encompassing this interval then gives the desired result without encroaching on time slots otherwise occupied. However, in attempting to synthesize signal functions which are absolutely time limited, one necessarily causes the corresponding spectrum functions to spread beyond any given finite limits, by virtue of the uncertainty principle of Fourier transform theory.

Certainly there are interesting communications problems where by virtue of spatial isolation the entire usable spectrum is, in effect, available at all times to each channel (e.g., the local telephone system). Effective bandwidth is then limited solely by transmission and noise properties of the medium or by practical problems in the design of arbitrarily wideband terminal equipment. However, this limit is not sharp, and leaves the bandwidth difficult to define in terms of a single numeric measure \( W \). Then basic results of communication theory, including the very cornerstone relation

\[
C = W \log \left(1 + \frac{S}{N}\right)
\]

become correspondingly difficult to interpret.

The type of problem for which the sophisticated theory is at once more clear in its interpretation and more useful in its application is one in which the usable spectrum is shared by frequency multiplexing among a large number of channels with each assigned to a different one of a set of contiguous sharply defined frequency bands. Any spillover in frequency then causes the channels to interfere with one another, just as spill over in time causes the individual time-contiguous signals on one channel to interfere with one another. One must seek signal functions or sets of signal functions which are simultaneously concentrated both in time and bandwidth. This paper is concerned with sets of
functions which are time-bandwidth concentrated and have certain desirable geometric properties. It is directed mainly at the communications problem just mentioned but may have some interesting implications for the many-target radar problem.

We present the results simply as a collection of theorems and proofs. Interpretations thereof in the way of practical applications would vary widely according to individual circumstances and are therefore left largely to the reader. The results, however, have been formulated with a particular model in mind and with the general intention of relating this model more closely to the actual systems one might like to represent with it.

The model has been discussed quite extensively in the literature (e.g., [1], [2], [3]). It constitutes a discrete M-nary channel on which the M independent equi-probable channel symbols are each represented by one of a configuration of M vectors (usually of equal length) in Euclidean n space. Once each T seconds, a vector is received which consists of the sum of one of these symbol vectors and a spherical Gaussian-distributed random noise vector. The detection is by minimum distance (maximum likelihood).

In this model, for every doublet (M, n) there is an "optimum" configuration (see Shannon, [1]). In particular, for any n and M = n + 1 there is the "simplex," while for any n and M = 2n there is the "biorthogonal" configuration. Our results are specifically formulated in terms of these configurations (and of the plain orthogonal configurations, which are not optimum but which lead trivially to the biorthogonal).

The use of such configurations is meaningful only if the vector model is valid. Shannon [1] points out the obvious correspondence between the model and an actual system wherein signaling is accomplished by using waveforms of electromagnetic energy which have nominal duration T and nominal bandwidth W with n = 2WT. He also warns his readers to invoke this correspondence at their own peril. We eliminate one possible source of that peril simply by assuming that n, W, and T are given numbers, not necessarily (although perhaps desirably) obeying the n = 2WT relation even approximately. Reduction of the remaining peril by reducing the spurious noiselike effects of intersymbol interference and interchannel interference is one end product of the results to follow.

The model requires that the noise be "white Gaussian," whereas the various spurious effects are unlikely to be so.

1 A "simplex" configuration consists of the position vectors of the vertices of a regular simplex centered at the origin in n space. The n + 1 vectors are equi correlated with cross correlation of any pair equal to -1/n. A "biorthogonal" configuration in n space consists of n equal length mutually orthogonal vectors with the negative of each. It could be termed "cross polytope" by analogy to the term "simplex," but this is obviously too jacking in euphony.

2 "White Gaussian" noise is defined as noise whose vector representation in the model has all its n orthogonal components given by independent zero mean Gaussian variates with equal variances. (The more usual notion of white Gaussian noise is sufficient but not necessary for this more restricted notion.) A process which is white Gaussian in this sense makes each event a spherical Gaussian n variate.

To the extent that they are not white Gaussian, effects must be kept negligibly small compared to whatever true noise or noise-like effects are white Gaussian and should, of course, be kept small in order to reduce the total noise.

Functions will be related to vectors in the following manner. A set of n linearly independent functions \( \{x_r(t)\} \) will be given. We may assume without loss of generality that they are orthonormal such that

\[
\int_{-\infty}^{\infty} x_r(t)x_s(t) \, dt = \delta_{rs} \quad \text{(Kronecker delta)}. 
\]

These will be the basis functions which define the n space; Any square integrable function \( s(t) \) may be represented as a vector in the space by the ordered n tuple of numbers

\[
\sigma_i = \int_{-\infty}^{\infty} s(t)x_i(t) \, dt. 
\]

Then the function

\[
s(t) = \sum_{j=1}^{n} \sigma_j x_j(t) 
\]

is the orthogonal projection of \( s(t) \) onto the n space; the remaining component \( s(t) - s(t) \) is orthogonal to n space and is "lost" in the representation of \( s(t) \).

For simplicity of presentation, we will assume for the remainder of the paper that the functions \( x_r(t) \) are identically zero outside the interval \((0, T)\). This illustrates the salient features of the more general and will require a somewhat briefer development. Among other simplifications, this permits a direct connection between the functions \( x_r(t) \) and the impulse response of \( n \) "sampling networks" in the receiver of an actual system. Thus by letting \( x_r(t) = h_r(T - t) \) (where \( h_r(t) \) is the impulse response of the \( j \)th sampling network, \( \sigma_j \) becomes the sampled value at time \( T \) of the output of the \( j \)th network when the input is \( s(t) \).

We are interested primarily in the case where \( s(t) \) is a sequence of signal waveforms,

\[
s(t) = \sum_{k} \delta_{s}(t - kT),
\]

and where \( \{\delta_{s}(t)\} \) is the set of M channel symbol waveforms as they appear at the inputs to the bank of receiving sampling networks. Thus the ordered n tuple of output sample values at \( t = T \) represents \( \delta_{s} \), while the \( n \) vector of sample values at \( t = (k + 1)T \) represents \( \delta_{s} \). Of course, if each of the \( \delta_{s} \) are not absolutely time-limited to \((0, T)\), then the sample values at \( t = (k + 1)T \) are not a "pure" representation of \( \delta_{s}(t - kT) \).}

3 For further simplification, we will assume either 1) the signaling is at baseband for the channel n question, with the channels occupying modulation bands which start just above baseband, or 2) that the signals are generated at baseband, mixed up to some assigned RF band, then coherently demodulated back to baseband after reception. The modulation-demodulation process is conceptually included in the channel, but all filtering which follows demodulation is assumed to be included as part of the sampling networks. These assumptions are necessary in that we may invoke certain prior results which apply only to baseband signals.
include contributions from the transient "tails" of earlier arrivals and the anticipatory buildups to later arrivals. This is one form of intersymbol interference. As mentioned above, however, we combine this with the noise and define the "pure" representation of $s_i(t)$ by

$$
\sigma_{i1} = \int_0^T s_i(t)x_i(t) \, dt
$$

$$
\sigma_i(t) = \sum_{j=1}^n \sigma_{ij}x_i(t).
$$

For lack of any better or more reasonably simple measure of the intersymbol interference, we will use a measure of relative energy concentration within $(0, T)$ of each of the $x_i(t)$. This will be most meaningful when the restriction of $s_i(t)$ to $(0, T)$ is in fact equal to $s_i(t)$ so that none of the energy of $s_i(t)$ within $(0, T)$ is lost in the projection. A similar measure of spectral energy concentration within $(-W, W)$ in the frequency domain will be used as a measure of interchannel interference.

If the signal that one must transmit in order to have $s_i(t)$ appear at the receiver has spectral energy components which "spill over" beyond the assigned baseband $(-W, W)$, then this will cause a noise-like interference to appear within the modulation bands assigned to other channels. Suppose we were to confine our own transmission entirely to the assigned band. Then, if our attitude for the other channel users was reciprocated in kind, we would have eliminated one form of interchannel interference within our own channel, viz., that which arises from other channels' transmission appearing in our demodulated wave within our assigned baseband $(-W, W)$. However, we have already assumed that the impulse responses of our sampling networks are absolutely time limited and therefore cannot be absolutely band limited. They must admit some energy from outside the $(-W, W)$ band, and our demodulation of the other channels' transmitted energy will place it in bands starting just outside $(-W, W)$. This latter form could be measured by the relative spectral energy concentration of the $x_i(t)$, but it is measured equally well and more conveniently by the concentration of the $s_i(t)$, since this more nearly represents the comparative disturbing effect as and when each different signal becomes the one whose presence one is attempting to detect. We will also assume by the above philosophy of reciprocation that the first kind of interchannel interference is estimated by the spectral concentration of our own transmitted signals, and that this in turn must be measured by the spectral concentration of the signals $i(t)$ which appear in the receiver (since the transmission characteristics of the medium are not specified). That is, we assume the other users will avoid our assigned spectral band approximately to the same degree that we avoid theirs.

With this background we may now proceed to the business at hand. In Section II, we begin with a brief review of some important recent results in the theory of time-bandwidth concentrated functions which will be the basis for our own results. Since the final results are simply set down in Section IV in the form of Theorem: Proof, with little intervening discussion, the results will be summarized and discussed beforehand in Section III.

II. Review of Background Theory

This paper is concerned with the extent to which sets of functions forming simplex and biorthogonal vector configurations in $n$ space can be simultaneously concentrated in a time interval of width $T$ and in a bandwidth interval $(-W, W)$. In particular, it seeks to explore the dependence of this concentration on $n$, $W$, $T$, and the particular $n$-dimensional function space in which the configuration is imbedded. Clearly this must be related in some way to the uncertainty principle of Fourier transform theory.

In a series of recent papers [4], [5], [6], Landau Pollak, and Slepian have shown that the prolate spheroidal wave functions (PSWF's) play a fundamental role in this uncertainty principle. Among the many significant results in this monumental work, they have at last provided a rigorous mathematical statement of the old engineering adage, that the space of functions time bandwidth concentrated in $WT$ is $2WT$ dimensional. In view of this, it would seem that the question of dimensionality in relating the vector model to actual channels, as discussed in Section I, is now properly answered. Further, the manner in which they provide this answer in terms of time-bandwidth concentration would seem to provide answers simultaneously to the interference problems.

Unfortunately, their results are formulated from a different point of view and in terms of subsets of the set of functions in $L^2$ rather than in terms of finite dimensional subspaces of $L^2$ itself. These subsets 1) in general may not contain any simplex set of $n + 1$ functions or orthogonal set of $n$ functions if $n = [2WT] + 1$ (one plus the largest integer in $2WT$), 2) almost surely will not contain any entire $n$-dimensional subspace if $n = [2WT] + 1$, and 3) in general may not contain any subspace of dimensionality greater than 1. (Any which contain no one-dimensional subspace must be empty.) Thus we must reformulate their results if we are to apply them to finite subspaces. The following brief review of their more pertinent results is provided for the reader's convenience.

We will be interested mainly in two infinite dimensional linear subspaces of $L^2$, which are the ranges of two operators whose domain is all of $L^2$. Norms are defined throughout to be the ordinary Hilbert norms in $L^2$. Following Landau, Pollak, and Slepian, we define the operator $D$ to be that of absolute time truncation to the interval $(-T/2, T/2)$, and operator $B$ to be that of
absolute bandwidth limiting to the interval \((-W, W)\). Their ranges are denoted \(\mathcal{D}\) and \(\mathcal{A}\), respectively. Landau, Pollak, and Slepian focus their attention mainly on functions in \(\mathcal{A}\), but for reasons previously explained we choose rather to concentrate on BD whose eigenvectors lie in \(\mathcal{D}\), rather than that of DB whose eigenvectors lie in \(\mathcal{B}\). As they point out, how-ever, our form follows directly from theirs by time-frequency duality.

The eigenvector solutions to

\[ \lambda f = DBf \]

are the time truncated PSWF’s \(\{D\psi_i(t; c)\}_n\) in which \(c\) is a parameter equal to \(\pi WT\), with eigenvalues \(\left\{\lambda_i(c)\right\}_n\); \(1 > \lambda_0 > \lambda_1 > \cdots > 0\). The \(\lambda_i(c)\) are monotone increasing functions of \(c\) for all \(i\). We normalize \(\|D\psi_i\|^2 = 1\), and hereafter drop the prefixed operator \(D\) such that the notation \(\psi_i\) implies time truncated PSWF’s. Then

\[ \langle \psi_i, \psi_j \rangle = \delta_{ij} \]

\[ \langle B\psi_i, B\psi_j \rangle = \lambda_i \delta_{ij}. \]

Relative frequency concentration \(C_s(f)\) of any \(f \in \mathcal{D}\) is defined by

\[ C_s(f) = \frac{\|f\|^{-2}}{\|Bf\|^2}. \]

The function \(\psi_i\) is the maximally concentrated function in \(\mathcal{D}\); \(\psi_i\) is the maximally concentrated of all functions orthogonal to \(\psi_i\) in \(\mathcal{D}\), etc.

In terms of the \(\{\lambda_i\}_n\), \(C_s(f)\) for arbitrary \(f \in \mathcal{D}\) is given by

\[ C_s(f) = \left[ \sum_{i=0}^{n} a_i^2 \right]^{-1} \left[ \sum_{i=0}^{n} \frac{a_i}{\lambda_i} \right], \]

in which the \(\{a_i\}_n\) are the coefficients of the Fourier expansion of \(f\) in the PSWF’s.

Landau, Pollak, and Slepian [4]-[6] prove the following two theorems (6), Theorems 1 and 3):

Theorem: Let \(\mathcal{B}(\varepsilon)\) be the subset of all functions \(f \in \mathcal{D}\) for which \(C_s(f) \geq 1 - \varepsilon\). Then for any \(N\) the first \(N + 1\) PSWF’s achieve the final minimum in

\[ \min_{\{a_i\}_n} \max_{f \in \mathcal{B}(\varepsilon)} \min_{\{a_i\}_n} \left\{ \frac{\|f\|^{-2}}{\|f - \sum_{i=0}^{n} a_i \phi_i\|^2} \right\}. \]

Theorem: For all \(f \in \mathcal{B}(\varepsilon)\) and \(N = [2WT]\),

\[ \min_{\{a_i\}_n} \frac{\|f\|^{-1}}{\|f - \sum_{i=0}^{N} a_i \phi_i\|^2} \leq \frac{\varepsilon}{1 - \lambda_{N+1}} \leq 12 \varepsilon. \]

The latter states that for all \(f \in \mathcal{B}(\varepsilon)\), a Fourier expansion in PSWF’s using \([2WT] + 1\) degrees of freedom is sufficient to represent \(f\) within a relative integrated square error of \(12\varepsilon\), and in this sense, the space of absolutely time-limited, nominally band-limited functions is \(2WT\) dimensional. (Similar results are shown for the more general case of nominally time-bandwidth-limited functions.)

However, in the context of the problem as outlined previously, \(n\) is a number already chosen, with the aim of determining \(n\) as one criterion but subject to other considerations such as the very space of representations to which we have decided to limit ourselves in the actual detection.

The pertinent question for the problem as stated is not, “What is the effective dimensionality of a subset of all functions time bandwidth concentrated to a given degree?”, but rather, “What is the effective concentration of a given configuration in a given \(n\)-dimensional linear subspace of \(L^2\)?” We will define this effective concentration to be the concentration of the least concentrated vector in the configuration when it is oriented to maximize this least concentration. This max-concentration criterion is used because, as shown below, the average concentration for each of the particular configurations considered herein is independent of orientation.

III. SUMMARY AND DISCUSSION

We are now prepared to summarize the results which follow. Let \(X_n\) be an \(n\)-dimensional linear subspace of \(\mathcal{A}\). Then it is shown (Theorem III) that the average of the concentrations of the vectors in any orthonormal set of \(n\) vectors in \(X_n\) is the same as for any other orthonormal set of \(n\) vectors in \(X_n\). From this it is shown (Corollary III-a) that the average of the concentration of any simplex set of \(n + 1\) vectors in \(X_n\) has the same value as for an orthonormal set.

Next \(\varepsilon\) is shown (Theorem IV) that for every \(n\), \(X_n \subset \mathcal{D}\), there exists at least one orthonormal basis of \(X_n\), each of whose vectors has concentration exactly equal to the average concentration. From this it follows trivially (Corollary IV-a) that there exists a biorthogonal set of \(2n\) vectors with the same property. This is clearly the max-concentrated biorthogonal configuration in \(X_n\). Next it is shown (Corollary IV-b) that simple configurations, each of whose vectors has concentration equal to the average, do not exist in general. However, there are special cases for which they do exist as noted, and the method is valid for “most” if not all other cases is outlined for finding simplex configurations whose least concentrated vectors very nearly achieve the average.

Configurations obtained by this method are believed to be maxi-minimally concentrated, but at present this is a pure conjecture.

Vectors in these maxi-minimally concentrated configurations are expressed as \(n\) tuples in what is den-
as a “proper” basis of $X_a$. This basis $\{x_i\}^n$ has the double orthogonality property

$$(x_i, x_j) = \delta_{ij}$$

$$\langle Bx_i, Bx_j \rangle = \mu_{ij} \delta_{ij}. $$

Such a basis is shown (Theorem I) to exist for all $X_a \subset \mathcal{D}$. In fact, $\{x_i\}^n$ and $\{\mu_{ij}\}^n_{i,j=1}$, respectively, are the normalized eigenvectors and eigenvalues of a certain operator on $X_a$, analogous to the operation $DB$ on $\mathcal{D}$ itself. The set of function $Tr(X_a)$ is defined as the trace of this operator on $X_a$, i.e., $Tr(X_a) = \sum \mu_i$. Therefore, the average concentration for any orthonormal basis of $X_a$ is $\bar{\mu} = 1/n \ Tr(X_a)$. For each given $X_a$, the $\mu_i$ and, therefore $\bar{\mu}$ are clearly monotone increasing functions of $\epsilon$ (the $x_i$ themselves depend parametrically on the parameter $\epsilon$) and are semi-ordered. $1 > \mu_0(\epsilon) > \cdots > \mu_n(\epsilon) > 0$ for all $\epsilon > 0$.

Thus at least for biorthogonal configurations, the quantity of interest for determining $\epsilon$ from given $WT$ and $X_a$ is $\bar{\mu}(\epsilon) = 1 - \epsilon^2$. In the special cases noted, this same quantity determines $\epsilon$ from given $WT$ and $X_a$, for simplex configurations. That is, in line with the notation of [4]–[6], we measure concentration by the parameter $\bar{\epsilon}$ using the relation max-min $C_\epsilon = 1 - \bar{\epsilon}^2$. For these configurations, max-min $C_\epsilon = \bar{\mu}$. For simplex configurations in general, max-min $C_\epsilon$ will equal a weighted average $\mu_i$ with weights fairly close to unity, and therefore $\bar{\mu}$ will be a good upper bound approximation to max-min $C_\epsilon$.

The following is suggested as a formal scheme for choosing $X_a$ and finding $\{x_i\}^n$ and $\{\mu_{ij}\}^n_{i,j=1}$ for it when $WT$ is given. First find $x_i(t)$ as the most $B$-concentrated unit energetic function in $\mathcal{D}$ which one is willing and able to implement. Next find $x_i(t)$ as the most $B$-concentrated unit energetic function orthogonal to $x_i$ in $\mathcal{D}$ which one is willing and able to implement. Continue in this manner to obtain $\{x_i\}^n$. Clearly, this spans a $X_a$ and by construction, it is in fact a proper basis for $X_a$. Except for sign changes, it is the unique proper basis if the concentrations of the $x_i$ are strictly ordered. Presumably, the numerical values of the $\mu_i$ will be obtained as a by-product of the calculations leading to the choice of the $\{x_i\}^n$. As a guide in determining how much effort it is profitable to expend on the implementation of the $\{x_i\}^n$, it is shown (Theorem II) that for any $X_a \subset \mathcal{D}$, $\mu_i(c) \leq \lambda_i(c), \mu_i(c) \leq \lambda_i(c), \cdots, \mu_i(c) \leq \lambda_i(c)$ for all $c$, where the $\lambda_i$ are the eigenvalues of $DB$ as noted previously. We might note, incidentally, and without proof herein that equality can hold for some value of $c$ and for some $i = k$ only if it holds for all $i < k$, in which case $x_i(t; c) = \psi_i(t; c)$ for all $i \leq k$ at that value of $c$.

In this work, we have made no attempt at a direct clarification of the significance of Landau and Pollak’s Theorem 3 [6] for the class of problems considered. We have, in fact, taken a directly opposite viewpoint in seeking to determine the extent to which a particular $n$-dimensional signal space (a linear subspace of $\mathcal{D}$) can be considered $WT$-limited in a particular problem, rather than seeking to determine the extent to which the entire subset, of functions in $\mathcal{D}$ which are $B$ concentrated by a given amount (not a linear subspace of $\mathcal{D}$) can be considered $2WT$ dimensional. Thus we virtually abandon the $a$ priori assumption that $n = 2WT$. Nevertheless, our results are for the most part a straightforward reformulation of Landau and Pollak’s from the alternate viewpoint and therefore retain a close relation to theirs.

When $n = 2WT$, one should expect that the task of finding and implementing a $X_a$ for which $1/n \ Tr(X_a)$ represents a high degree of concentration should prove relatively easy. As $n$ is decreased below $2WT$, it should become progressively easier. Conversely, if $n = 2WT/(1-\epsilon)$, then $1/n \ Tr(X_a) < 1 - \epsilon^2$ for all $X_a \subset \mathcal{D}$. This follows as a direct consequence of Corollary II-a which states $Tr(X_a) \leq Tr(\Psi) < Tr(\mathcal{D}) = 2WT$.

The numerical significance of the differences between the two approaches may be illustrated by some examples taken from Table I. One might, for example, require that all signal functions be at least 90 per cent concentrated. But clearly from the table, $\Theta(\sqrt{0.1})$ for $2WT = 2.55$ contains only two mutually orthogonal vectors and for $2WT = 5.10$ contains only five mutually orthogonal vectors, whereas, $[2WT] + 1$ equals 3 and 6 respectively, for the two cases. Unfortunately, data are not available on the behavior of $\lambda_i(c)$ for larger $n$ and larger $c = \epsilon WT$, but it appears quite possible that $\Theta(\sqrt{0.1})$ for large $2WT$ contains more than $[2WT] + 1$ mutually orthogonal vectors. An upper bound is $[2WT/0.9]$, and it is believed that this upper bound is approached asymptotically with increasing $2WT$. Thus if one is interested in orthogonal sets of signal functions $WT$-limited to a prescribed degree, $[2WT] + 1$ is no more than an estimate of the total number of such functions theoretically available. The approach adopted herein can be used to determine the actual number available in practice. It is interesting also to note that the space spanned by the orthogonal set can contain a vector considerably less concentrated than the vectors in the set. For example, $\Theta(\sqrt{0.1})$ for $2WT = 5.10$ contains five mutually orthogonal vectors whose actual concentrations are all equal to 0.94*, yet the 5 space they span contains a vector (not in $\Theta(\sqrt{0.1})$) whose concentration is less than 0.75.

* Data for this table are taken from Siegel and Pollak [4], Table I.
Thus far we have considered only an optimization with respect to the receiver representations \( \{s_i\} \) of the received signals. Clearly, if the actual received signals are themselves virtually time-limited (no intersymbol interference), then they are identical to the \( \{s_i\} \). In this case, both types of interchannel interference described in the introduction are mini-maxed. If, alternatively, the actual received signals are virtually band-limited (no interchannel interference of the first type) then it is shown (Corollary I-a) that for every element \( x \in X_\alpha \) there exists a unique element \( z \in B \) such that \( Dz = x \). From this and Theorem III it follows (Corollary III-b) that for any orthonormal set \( \{s_i(t)\}_1^n \) of \( X_\alpha \), the average reciprocal time concentration \( C_{2D}(\delta) \) of the corresponding set \( \{\delta_i(t)\}_1^n \) is the same as for any other orthonormal basis of \( X_\alpha \), where \( C_{2D}(f) = \|f\|^{-2} ||Df||^2 \). The same is shown (Corollary III-c) to be true of any simplex set of \( n + 1 \) vectors in \( X_\alpha \). Finally it is shown (Corollary IV-c) that for any \( n \) and any \( X_\alpha \subset D \) there exists an orthonormal basis \( \{\delta_i(t)\}_1^n \) for which

\[
\min_i C_{2D}(\delta_i) = C_{2D}(\delta)^{-1},
\]

and that this achieves the max-min \( C_{2D}(\delta_i) \) over all orthonormal bases of \( X_\alpha \). In this paper, we do not attempt either to prove or disprove the existence in general of an orthonormal basis which simultaneously achieves max-min \( C_{2D}(\delta_i) \) and max-min \( C_{2D}(\delta_i) \). However, interesting special cases in which they are achieved simultaneously are noted. Similar special cases lead to simultaneous achievement of both max-mins for simplex configurations.

With regard to the more general case of transmitted signals which are neither virtually time-limited nor virtually band-limited, little can be said in line with the above. The signals \( s_i \) for which \( Ds_i = s_i \) are no longer unique. One might investigate transmitted signals from the space \( D + B \), such as \( a s_i + (1 - a) s_i \), where \( s_i \in B \) as in the preceding paragraph and \( 0 \leq a \leq 1 \). For specific problems this may indicate a desirable compromise between no intersymbol interference and no interchannel interference of the first type. However, a sufficiently accurate model of a given physical situation may include a choice of transmitter functions from \( B \).

A theory sufficiently general to cover this case is beyond the scope of the present effort.

IV. DETAILED RESULTS

Theorem I: For each \( X_\alpha \subset D \), there exists a basis \( \{x_i(t)\}_1^n \) with the double orthogonality property \( (x_i, x_j) = \langle Bx_i, Bx_j \rangle = \mu_i \delta_{ij} \). (It is termed the “proper basis.”)

Proof: The proof follows trivially from the observation that any projection operator \( P \) is characterized by

\[
P^2 = P \quad (P \text{ is idempotent}).
\]

\( B \) is such a projection (of \( \mathcal{L}_2^n \) onto \( B \)). Define \( Fx \) as a projection of \( \mathcal{L}_2^n \) onto \( X_\alpha \) and note that \( P_{Bx} = x \) for \( x \in X_\alpha \). Then the combined operator \( P_{Bx}B \) is continuous with \( n \) independent solutions to

\[
\mu x = P_{Bx}Bx.
\]

The eigenvectors associated with unequal eigenvalues are orthogonal, and those associated with equal eigenvalues can be chosen as orthogonal. All can be normalized in \( \mathcal{L}_2^n \). Denote these eigenvectors and eigenvalues by \( \{x_i(t)\}_1^n \) and \( \{\mu_i\}_1^n \), respectively. Then

\[
(Bx_i, Bx_j) = (B^2x_i, x_j) = (Bx_i, x_j) = (Bx_i, P_{Bx}x_i) = (P_{Bx}Bx_i, x_j) = \mu_i \delta_{ij}.
\]

Corollary I-a: For every element \( x \in X_\alpha \) there is a unique element \( \bar{x} \in B \) such that \( D\bar{x} = x \). Moreover, \( \bar{x} \in Bx_\alpha \).

Proof: Uniqueness follows from the fact that each element \( f(t) \in B \) is an entire function of time therefore completely determined by its behavior \(-T/2, T/2\) or any other finite interval. Existence is shown by construction. Note that the operators \( P \) and \( D \) are naturally ordered \( P \leq D \) which implies \( P_B = D P = D \). Since \( x \in X_\alpha \), \( P_{Bx} = x \). Therefore \( D\bar{x} = x \) which implies \( P_{Bx}x \). Let the expansion of \( \{x_i(t)\}_1^n \) be given by \( x = \sum_i^\alpha a_i x_i \). Then \( \bar{x} = \sum_i^\alpha (a_i/\mu_i) \). Clearly satisfies \( P_{Bx} = x \), and indeed \( \bar{x} \in Bx_\alpha \).

Theorem II: For any \( X_\alpha \subset D \), the eigenvalues \( \mu(x) \) of \( Px \) are majorized by the first \( n \) eigenvalues \( \Lambda(x) \) of \( DB \). That is, \( \mu(x) \leq \Lambda(x) \).

\[
\mu(x) \leq \Lambda(x), \quad \mu(x) \leq \Lambda(x), \ldots, \quad \mu(x) \leq \Lambda(x).
\]

See Landau and Pollak [6], p. 1298. See also Dunford and Schwartz [7], p. 480. Dunford and Schwartz define a more general projection (nonorthogonal) which does not require self-adjointness. However, the present work deals only with orthogonal projections.
**Proof:** This theorem in more general form was apparently first proved by Weyl [8]. The following simple proof of the above form is given for the reader's convenience. Let \( \Psi \subseteq \mathcal{D} \) be the space spanned by the first \( n \) PSWF's \( \{ \psi_i \} \) and let \( P \) be the projection of \( \mathcal{D} \) onto \( \Psi \). Let \( \tilde{\psi}_i \) denote \( P \tilde{\psi}_i \) where \( \tilde{\psi}_i \in \{ \psi_i \} \). Then either (Case 1) \( \{ \tilde{\psi}_i \} \) is an independent set, or (Case 2) it is a dependent set, i.e., \( \{ \tilde{\psi}_i \} \) either spans or does not span \( \Psi \).

Thus, **Case 1:** There exists a unique linear combination of the \( \tilde{\psi}_i \) satisfying

\[
\sum_i a_i \tilde{\psi}_i = a \psi_{n-1}
\]

\[
\sum_i a_i^2 = 1; \quad 0 < \alpha \leq 1.
\]

The vector \( \psi = \sum_i a_i \psi_i \) is a unit energetic vector in \( X_a \). Since \( \{ \psi_i \} \) is complete in \( \mathcal{D} \) and \( X_a \subseteq \mathcal{D} \), \( \psi \) may be expanded in a Fourier series in the \( \{ \psi_i \} \). This yields

\[
x = \alpha \psi_{n-1} + \sum_{i=1}^n \alpha_i \psi_i,
\]

\[
a^2 + \sum_{i=1}^n \alpha_i^2 = 1.
\]

From this

\[
C_B(x) = a^2 \lambda_{n-1} + \sum_{i=1}^n \alpha_i^2 \lambda_i \leq \lambda_{n-1}.
\]

However, by virtue of the double orthogonality of \( \{ \psi_i \} \),

\[
C_B(x) = \sum_{i=1}^n \alpha_i^2 \mu_i \geq \mu_n
\]

\[
: \mu_n \leq \lambda_{n-1}.
\]

**Case 2:** There exists at least one linear combination of the \( \tilde{\psi}_i \) for which

\[
\sum_{i=1}^n a_i \tilde{\psi}_i(t) = 0
\]

\[
\sum_i a_i^2 = 1.
\]

The vector \( \psi = \sum_i a_i \psi_i \in X_a \subseteq \mathcal{D} \). Therefore

\[
x = \sum_{i=1}^n \alpha_i \psi_i,
\]

\[
\sum_i \alpha_i^2 = 1.
\]

\[
C_B(x) = \sum_{i=1}^n \lambda_i \alpha_i^2 \leq \lambda_{n-1}
\]

\[
C_B(x) = \sum_{i=1}^n \alpha_i^2 \mu_i \geq \mu_n
\]

\[
: \mu_n \leq \lambda_{n-1}.
\]

In either case, \( \mu_n \leq \lambda_{n-1} \). Now let \( X_{n-1} \) denote the space spanned by \( \{ x_i \} \) and \( \Psi_{n-1} \) denote the space spanned by \( \{ \psi_i \} \). Repeat the above argument to show \( \mu_{n-1} \leq \lambda_{n-2} \). Repeat again for \( \{ x_i \} \). The final statement \( \mu_1 \leq \lambda_0 \) is obvious since \( \psi_0 \) is the most concentrated vector in \( \mathcal{D} \).

Q.E.D.

We now define the set function \( \text{Tr}(X_n) \) to be the trace of the operator \( P \tilde{\psi}B \) on \( X_n \). That is, \( \text{Tr}(X_n) = \sum_i \mu_i \). It follows from Theorem II that \( \text{Tr}(X_n) \leq \text{Tr}(\Psi) \). Clearly \( \text{Tr}(\Psi) \leq \text{Tr}(\mathcal{D}) = \sum_i \lambda_i \), since \( \lambda_i > 0 \) for all \( i \). It is also clear from the results of [4], [5] and [6] that \( \text{Tr}(\mathcal{D}) \) (which is the trace of \( DB \) on \( \mathcal{D} \)) equals the trace of \( BD \) on \( \mathcal{D} \). The operation \( BDf \) is defined as

\[
BDf(t) = \int_{-\pi/2}^{\pi/2} \sin 2\pi W(t-t') f(t') \ dt'
\]

from which

\[
\text{Trace } BD = \int_{-\pi/2}^{\pi/2} \lim_{t \to t'} \frac{\sin 2\pi W(t-t')}{\pi(t-t')} dt'
\]

\[
= 2\pi W.
\]

We have thus proven:

**Corollary II-a:** \( \text{Tr}(X_n) \leq \text{Tr}(\Psi) < \text{Tr}(\mathcal{D}) = 2\pi W \) for all \( X_n \subseteq \mathcal{D} \).

We next state and prove:

**Theorem III:** Given any orthonormal basis \( \{ s_i(t) \} \) for \( X_n \subseteq \mathcal{D} \), the average concentration \( C_B(s_i) \) exactly equals \( 1/n \text{Tr}(X_n) \) and is therefore the same for all orthonormal bases of given \( X_n \).

**Proof:** Each of the functions \( s_i \in \{ s_i \} \) may be expressed as

\[
s_i(t) = \sum_{i=1}^n \sigma_i s_i(t)
\]

wherein

\[
\sum_{i=1}^n \sigma_i s_i(t) = \delta_{ij}.
\]

Let \( S \) denote the \( n \times n \) matrix \( \{ \sigma_{ij} \} ; i, j = 1, 2, \ldots, n \). \( S \) is orthogonal by definition. Now let \( M \) denote the \( n \times n \) matrix \( \{ \mu_{ij} \}; i, j = 1, 2, \ldots, n \) wherein

\[
\mu_{ij} = \begin{cases} 0, & i \neq j \\ \mu_i, & i = j \end{cases}
\]

Clearly \( \text{Trace } M = \text{Tr}(X_n) \). The transformation

\[
\text{Smoke}^t = S
\]

is a similarity transform and therefore trace invariant. Note that the diagonal element \( c_{ii} \) (\( i = 1, 2, \ldots, n \))
of $C$ is given by

$$c_{ii} = \sum_{j=1}^{n} \sigma_{ij} \mu_j = C_B(s_i).$$

Therefore

$$\text{Trace } C = \text{Tr} (X_n) = \sum_{i=1}^{n} C_B(s_i)$$

or

$$C_B(s_i) = \frac{1}{n} \text{Tr} (X_n). \quad \text{Q.E.D.}$$

Corollary III-a: Given any $X_n \subset \mathbb{D}$ and any simplex set of $n+1$ equi-energetic equicorrelated vectors $\{v_i(t)\}_{i=1}^{n+1}$, the average concentration

$$C_B(v_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} C_B(v_i)$$

exactly equals $1/n \text{Tr} (X_n)$ and is therefore the same for all simplex sets in the given $X_n$.

Proof: We assume without loss of generality that the $v_i$ are unit energetic. Then each of the functions $v_i \in \{v_i\}_{i=1}^{n+1}$ may be expressed as

$$v_i(t) = \sum_{j=1}^{n} \xi_{ij} x_j(t)$$

wherein (from the definition of simplex set)\textsuperscript{1}

$$\sum_{j=1}^{n} \xi_{ij} \xi_{jk} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Let $V$ denote the $(n+1) \times n$ matrix $(\xi_{ij})$. Then form $\Gamma = VMV^t$ where $M$ is the diagonal $n \times n$ matrix $(\mu_i)$ as above. Note that the diagonal element $\gamma_{ii}$ ($i = 1, 2, \ldots, n+1$) of $\Gamma$ is given by

$$\gamma_{ii} = \sum_{j=1}^{n} \xi_{ij}^2 \mu_j = C_B(v_i).$$

Now from $V$ form the $(n+1) \times (n+1)$ orthogonal matrix $\tilde{V}$ by adding an $(n+1)$th column all of whose elements are $+1/\sqrt{n}$, (thus making the row vectors mutually orthogonal), then scaling by $\sqrt{n/(n+1)}$ (to renormalize). Also form the $(n+1) \times (n+1)$ diagonal matrix $\tilde{M}$ as the direct sum of $M$ and the $1 \times 1$ matrix $(\mu)$ where $\mu = 1/n \text{Tr} (X_n)$. Then trace $\tilde{M} = (n+1)/n \text{Tr} (X_n)$, and the similarity transform $\tilde{V}M\tilde{V}^t = \hat{\Gamma}$ is trace invariant. Note that the diagonal element $\gamma_{ii}$ of $\Gamma$ is now given by

$$\gamma_{ii} = \frac{n+1}{n} \sum_{j=1}^{n} \xi_{ij}^2 \mu_j + \frac{1}{n+1} \mu$$

$$= \frac{n}{n+1} C_B(v_i) + \frac{1}{n+1} \mu.$$

Therefore

$$\gamma_{ii} = \frac{n+1}{n} \sum_{j=1}^{n} \xi_{ij}^2 \mu_j + \frac{1}{n+1} \mu$$

$$= \frac{n}{n+1} C_B(v_i) + \frac{1}{n+1} \mu.$$

Then

$$\text{Trace } \hat{\Gamma} = \frac{n+1}{n} \text{Tr} (X_n) = \frac{n}{n+1} \sum_{i=1}^{n+1} \gamma_{ii}$$

$$= \frac{n}{n+1} \sum_{i=1}^{n+1} C_B(v_i) + \frac{1}{n} \text{Tr} (X_n).$$

Therefore

$$\frac{n}{n+1} \sum_{i=1}^{n+1} C_B(v_i) = \text{Tr} (X_n).$$

or

$$C_B(v_i) = \frac{1}{n} \text{Tr} (X_n). \quad \text{Q.E.D.}$$

Corollary III-b: Given any orthonormal basis $\{s_i(t)\}_{i=1}^{n}$ for $X_n$, denote by $\{\hat{s}_i(t)\}_{i=1}^{n}$ the unique set in $BX_n$ for which $D\hat{s}_i = s_i$. Then the average reciprocal time concentration

$$C_B^{-1}(\hat{s}_i) = \frac{1}{1/n \text{Tr} (X_n)} \sum_{i=1}^{n} C_B^{-1}(s_i)$$

exactly equals the average reciprocal eigenvalue $\mu^{-1}$, and is therefore the same for all orthonormal bases of $X_n$.

Proof: By definition,

$$C_B^{-1}(\hat{s}_i) = || D\hat{s}_i ||^{-2} = || s_i ||^{-2} = || s_i ||^{-2} = || s_i ||^2$$

since $|| s_i ||^2 = 1$. But from Theorem I,

$$|| s_i ||^2 = \sum_{i=1}^{n} \sigma_{ii} s_i$$

Now denote by $\hat{M}$ the diagonal matrix $(1/\mu)^{11}$ and $\hat{C}$ the similarity transform $S\hat{M}\hat{S}^t = \hat{C}$. Note that the diagonal element $\hat{\gamma}_{ii}$ of $\hat{C}$ is given by

$$\hat{\gamma}_{ii} = \sum_{j=1}^{n} \frac{\sigma_{ii}^2}{\mu_j} = || \hat{s}_i ||^2.$$

Therefore

$$|| \hat{s}_i ||^2 = \frac{1}{n} \text{Trace } \hat{C} = \frac{1}{n} \text{Trace } \hat{M} = \frac{1}{\mu^{-1}}. \quad \text{Q.E.D.}$$

Corollary III-c: Given any simplex set $\{v_i(t)\}_{i=1}^{n+1}$ for $X_n$, the average reciprocal time concentration

$$C_B^{-1}(\hat{v}_i)$$

equals the average reciprocal eigenvalue $\mu^{-1}$ and is therefore the same for all simplex sets in $X_n$.

Proof: Following the proofs of III-a and III-b, note that the diagonal element $\gamma_{ii}$ of $\hat{\Gamma} = VMV^t$ is given by

$$\gamma_{ii} = \sum_{j=1}^{n} \xi_{ij}^2 \mu_j = C_B^{-1}(v_i).$$

$\hat{M}$ is in fact $M^{-1}$.\textsuperscript{11}
Form $\hat{\mathbf{F}}$ as above, and $\hat{\mathbf{M}}$ as the direct sum of $\mathbf{M}$ and the $1 \times 1$ matrix $\left(\mu^2\right)$. Then as above,

$$\text{Trace } \hat{\mathbf{F}} = \frac{n}{n+1} \sum_{i=1}^{n+1} \mathbf{C}_D^i(\psi_i) + \frac{1}{n} \text{Trace } \hat{\mathbf{M}}$$

$$= \text{Trace } \hat{\mathbf{M}} = \frac{n+1}{n} \text{Trace } \hat{\mathbf{M}}.$$

Therefore,

$$\mathbf{C}_D^{-1}(\psi_i) = \frac{1}{n} \text{Trace } \hat{\mathbf{M}} = \frac{1}{\mu^2}. \quad \text{Q.E.D.}$$

We have proven that simplex and orthogonal (and therefore, as we shall see, biorthogonal) configurations, of $n+1$ and $n$ (and $2n$) vectors, respectively, in a given $X_n$, all have the same average concentration. One might advance heuristic geometric arguments to show that any configuration with sufficient symmetry would have this property; indeed that any optimum code configuration as defined in [1] or [3], even those with such misfit values of $F$ and $n$ that symmetry is totally lacking, would have "very nearly" this property. That is, the average concentration is at most weakly dependent on orientation of the configuration. We therefore adopt a max-minimal criterion, and proceed to show how one can find maximally concentrated simplex and biorthogonal configurations.

Theorem IV: Given any $X_n \subset \mathbb{D}$, there exists at least one orthonormal basis $\{s_i(t)\}$ such that $\mathbf{C}_b(s_i) = 1/n \text{Tr } (X_n)$ for all $i = 1, 2, \ldots, n$.

Proof: We have only to prove that there exists an $n \times n$ orthogonal $S$ matrix such that the similarity transform $\mathbf{S}\mathbf{M}\mathbf{S}^T = \mathbf{C}$ yields $c_{ij} = c_{ji}$ for all $i, j = 1, 2, \ldots, n$. The following constructive proof was suggested by Landau. Consider the pair of row vectors ($n$ tuples) in the proper basis $\{x_i\}$:

$$\left(0, 0, \cdots, 0, 0, + \frac{\mu - \mu_n}{\mu_1 - \mu_n}, 0, 0, \cdots, 0, 0, + \frac{\mu_1 - \mu}{\mu_1 - \mu_n}\right)$$

These are mutually orthogonal and unit energetic. The first has concentration

$$\left(\frac{\mu - \mu_n}{\mu_1 - \mu_n}\right)\mu_1 + \left(\frac{\mu_1 - \mu}{\mu_1 - \mu_n}\right)\mu_n = \mu = \frac{1}{n} \text{Tr } (X_n).$$

The second has concentration

$$\left(\frac{\mu_1 - \mu}{\mu_1 - \mu_n}\right)\mu_1 + \left(\frac{\mu - \mu_n}{\mu_1 - \mu_n}\right)\mu_n = \mu_1 + \mu_n - \mu.$$

The entire process is then repeated to form

1) An $(n - 1)$ tuple in the basis $\{x_i\}_{1}^{n-1}$ which is unit energetic, has concentration $1/(n - 1) \text{Tr } (X_{n-1}) = 1/n \text{Tr } (X_n)$, and becomes an $n$ tuple in the original basis $\{x_i\}_{1}^{n}$ orthogonal to the one previously found. It constitutes the second-row of $S$.

2) A new proper basis $\{x_{n-1}\}_{1}^{n}$ for which $\text{Tr } (X_{n-1}) = (n - 2)/(n - 1) \text{Tr } (X_{n-1}) = (n - 2)/n \text{Tr } (X_n)$.

3) A proper basis $\{x_i\}_{1}^{n-2}$ for $X_{n-2}$ with concentrations $\{\mu_i\}_{1}^{n-2}$ for which $\text{Tr } (X_{n-2}) = 1/n \text{Tr } (X_n)$.

The process is repeated $n - 3$ more times, thereby generating a total of $n - 1$ $n$ tuples (rows of $S$) which are all mutually orthogonal, unit energetic, and have concentrations $1/n \text{Tr } (X_n)$ for each of which $\mathbf{C}_b(s_i) = 1/n \text{Tr } (X_n)$.

Corollary IV-a: Given any $X_n \subset \mathbb{D}$, there exists an equi-energetic biorthogonal code set of $2n$ vectors in $X_n$, each of which has concentration $1/n \text{Tr } (X_n)$.

Proof: The proof is trivial. We need only form the set of $2n$ vectors $\{\pm x_i\}_{1}^{n}$ from $\{s_j\}_{1}^{n}$ just found and note that $\mathbf{C}_b(s_i) = \mathbf{C}_b(-s_i) = \mathbf{C}_b(s_i)$.

Corollary IV-b: It is not true in general that given an $X_n$ there exists a simplex set of $n + 1$ code vectors in $X_n$, each of which has concentration $1/n \text{Tr } (X_n)$.

Proof: To prove the corollary as stated, we need only cite a single example to contradict the converse. We
choose the trivial example of a "simplex" (triangle) in $X_3$. The code set consists of three vectors in the plane, separated by $120^\circ$. Given any $\mu_1, \mu_2$ such that $1 > \mu_1 > \mu_2 > 0$, the orientation which maximizes the least concentration is that which places one of the vectors along the $x_1$ axis. Then the three concentrations are $\mu_1$, $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$, and again $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Thus the maxi-minimal concentration is $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 < \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = \frac{1}{2} \text{Tr} (X_3)$. (Of course if $\mu_1 = \mu_2$, then every vector in the space has concentration $\frac{1}{2} \text{Tr} (X_3)$, but this does not occur in general).

Q.E.D.

The corollary is intended to imply the existence of a wide class of cases for which a maxi-min equal to the average can be achieved, and a correspondingly wide or wider class for which it cannot. Unfortunately, the proof given contributes little to the implication. Following the proof of Corollary III-a, one can easily show that a necessary and sufficient condition for the existence of an equiconcentrated simplex code in $X_k$ is the existence of an orthogonal $(n + 1) \times (n + 1)$ matrix $\mathbf{P}$ having the same properties as those of the matrix $S$ in Theorem IV (but not necessarily constructed by the method given in the proof of Theorem IV). That is, to be a $S$ matrix it must contain a column all of whose elements are $+1/(\sqrt{n} + 1)$, while to be simultaneously an $S$ matrix from Theorem IV for the ficticious space $X_{n+k}$, for which the eigenvalues of $P \mathbf{Z}B$ are $\rho \cup \{p_i\}$, it must equalize the diagonal elements of $\mathbf{P} \mathbf{Z} \mathbf{P}' = \mathbf{I}$; however, this merely translates the problem to that of finding necessary and sufficient conditions for the existence of such a $S$, and these are not known. A sufficient condition, independent of the actual eigenvalues $\{p_i\}$, or of the actual space $X_k$, itself, is the existence of a Hadamard matrix $H$ of order $n+1$. (See Paley [9] and Peterson [10].)

A Hadamard matrix is a square matrix all of whose elements are $\pm 1$, and whose row vectors are mutually orthogonal. It remains Hadamard if the signs of all elements in any row or column are changed. Thus it may always be transformed to have all plus elements in the final column. If a Hadamard matrix $H$ of order $n + 1$ exists, then $1/(\sqrt{n} + 1)$ times the $H$ in this form will always yield the desired $\mathbf{P}$. (The proof is by inspection).

Another set of sufficient conditions is that the set $\{p_i\}$ of eigenvalues of $P \mathbf{Z}B$ on $X_n$ contain one value $\mu_i = \mu$, and that there exist a Hadamard matrix of order $n$. Cast the Hadamard matrix in a form such that the $j$th column has all elements negative. Then scale by $n^{1/2}$ and replace the $j$th column by a column of all whose elements are $-1/n$. Finally add an additional row vector containing all zeros except for a $+1$ in this $j$th column. This yields an $(n + 1) \times n$ $V$ matrix directly, and the reader may easily verify that $VMV' = \mathbf{I}$ with $\gamma_i = 1/n \text{Tr} (X_n)$ for all $i = 1, 2, \ldots, n + 1$.

If the second but not the first condition is satisfied, i.e., if $\mu_1 > \mu > \mu_2$, then this same manipulation on the $j$th or $(j + 1)$th column of $H$ yields a $V$ matrix for $X_n$ whose least concentrated vector has concentration $\rho - (\mu_1 - \mu)$ and $\mu_2$, respectively. The former cannot be improved by small perturbations in the orientation of the configuration. The latter can, however, be improved by applying to the $V$ matrix the simple rotation operation

$$R = \begin{bmatrix} I_i & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & I_{n-1,i-2} \end{bmatrix}$$

in which $I_i$ is the identity matrix of order $i$, and $\theta$ is given by

$$\sin^2 \theta = \rho - 2 \sqrt{1 + \frac{n^2}{n(n+1)} \mu - \frac{n^2}{n(n+1)} + \frac{1}{4}}$$

Then $V_{(new)} = V_{(old)} R$ contains $n/2 + 1$ row vectors, with concentrations all equal and slightly below $\mu$, the remaining $n/2$ row vectors have concentrations equal and slightly above $\mu$. The lower value is

$$C_min = \mu - 2(\mu - \mu_{i+1})$$

The greater of this number and $\frac{1}{n} \text{Tr} (X_k)$ is believed to be the maxi-minimal concentration for simplex codes in $X_k$. Similar but increasingly complex procedures may be used to derive simplex codes whose least concentration vector is "almost" $1/n \text{Tr} (X_k)$ in spaces $X_k$ of dimension one, two, or more above the order of a Hadamard matrix.

It should also be noted that the use of Hadamard matrices leads to a considerably simplification of procedure for constructing the $S$ matrices of Theorem IV. The matrix $1/\sqrt{n}$ times any Hadamard matrix of order $n$ is clearly an $S$ matrix, and increasing simple procedures may be used to derive simplex codes whose least concentration vector is "almost" $1/n \text{Tr} (X_k)$ in spaces $X_k$ of dimension one, two, or more above the order of a Hadamard matrix.

13 $n$ is even, since by hypothesis there exists an $H$ matrix of order $n$.

14 This fact was apparently noted concurrently by the present writer and by Petrich [11] working independently. However Petrich overlooked the generalizations to other dimensions and to the simplex sets.
The task is to find an $S$ matrix in the basis $\{x_i^{(j)}\}_{i=1}^n$ for the space $X_{n-j}$. Then $1/(\sqrt{n}-j)$ times any Hadamard matrix of order $n-j$ is such an $S$ matrix, and its row vectors expressed as $n$ tuples in the original basis $\{x_i\}_i^n$, along with the $j$ $n$ tuples already found, comprise the rows of the desired $S$ matrix for the original $X_n$. Rules for generating Hadamard matrices, of virtually all orders for which they are known to exist, are given by Paley [9].

The existence of Hadamard matrices of order $n$ leads to interesting consequences of a further corollary to Theorem IV. We first state and prove the corollary.

**Corollary IV-c:** For every $X_n \subset \mathcal{D}$ there exists at least one orthonormal basis $\{s_i(t)\}_{i=1}^r$ (with the corresponding pre-image set $\{s_i(t)\}_{i=1}^r$ in $BX_n$) for which

$$\min_i C_D(s_i) = \overline{C_D(s_i)}^{-1},$$

and this achieves the maxi-min $C_2(s_i)$ over all possible orthonormal bases of $X_n$.

**Proof:** The proof directly follows the construction in the proof of Theorem IV, with $\mu_j$ replaced by $\mu^{-1}$ for all $j$ and $\mu$ replaced by $\mu^{-1}$. That construction determines for any given positive definite $n \times n$ diagonal matrix $A$, with diagonal elements ordered either nonincreasing or nondecreasing, an orthogonal matrix $S$ such that the diagonal elements of $SAS^T = C$ are equalized. The $S$ so constructed for $A = \overline{M}$ has row vectors $s_i$ for each of which $C_D^r(s_i) = \overline{C_D^r(s_i)}$. This proves the existence. We prove that this achieves the maxi-min by contradiction. Assume there exists an orthonormal basis $\{s_i^{(j)}\}_{i=1}^r$ for $X_n$ such that

$$\min_i C_D(s_i^{(j)}) > \overline{C_D(s_i^{(j)})}^{-1}.$$  

Then

$$\max_i C_D^{-1}(s_i^{(j)}) < \overline{C_D^{-1}(s_i^{(j)})}.$$  

However it is clearly impossible for the largest of any set of real numbers to be less than the average. Q.E.D.

Now note that if there exists a Hadamard matrix of order $n$, the $S$ matrix obtained from it simultaneously equalizes the diagonal elements of $SMS^T = C$ and of $C^{-1}S^{-T}C^T$, and therefore simultaneously achieves maxi-min $C_D(s_i)$ and maxi-min $C_D^{-1}(s_i)$ for any given $X_n$ in $\mathcal{D}$. Similarly, if there exists a Hadamard matrix of order $n+1$, then the $\overline{V}$ matrix obtained from it specifies a simplex set $\{v_i^{(j+1)}\}_{i=1}^r$ in any given $X_n$ such that the two concentrations are simultaneously maxi-minimized. It is significant to note that since $M = n+1$ for simplex and $M = 2n$ for biorthogonal encodings, Hadamard matrices of the appropriate orders for both types of code will exist whenever $M$ is a power of 2.

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**References**


