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A NOTE ON THE TRANSIENT DEVELOPMENT OF A FREE SURFACE

by

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The Secretary,
The Institution of Civil Engineers.

DEAR SIR,

TRANSIENT DEVELOPMENT OF THE FREE SURFACE IN A HOMOGENEOUS EARTH DAM, by Suresh P. Brahma and Milton E. Harr, *Géotechnique* 12:4:283–302

There are several points in the above Paper which the Writer feels require discussion. The Authors use as their governing differential equation the linearized version of the Dupuit-Forchheimer or Boussinesq equation:

$$\frac{\partial h}{\partial t} = \frac{k\bar{h}}{n}\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and cite as their reference source Muskat's "Flow of homogeneous fluids". Muskat, however, goes to great lengths to point out the shortcomings of this equation (pp. 359–365). In the derivation the assumption is made that the slope of the free surface is everywhere small, and to linearize the resulting differential equation requires the additional assumption that h be not too different from $\bar{h}$. For the class of problems considered by the Authors neither of these assumptions holds.

Of greater importance, however, is the fact that in the derivation h is not a "head" but merely the height of the free water surface measured up from some datum. The vertical co-ordinate is eliminated as a consequence of the assumption that the vertical velocity component is zero. Hence the co-ordinates x, y which appear in the equation lie in a horizontal plane and not in a vertical plane, as supposed by the authors.

The "exact" theory for this problem includes Laplace's equation as the governing differential equation, together with an *a priori* unknown boundary location as part of the boundary conditions. The purpose of the approximations described is to allow the recasting of this fairly complicated problem into a much simpler form whereby the unknown boundary location itself becomes the variable in the differential equation. Hence, once these approximations are made nothing further needs to be said about the free surface, so that the first phrase of condition (3c) on p. 284 appears to be redundant.

In the formulation of the boundary value problems the condition (6b) on p. 285 implies that $h = H$ everywhere along the rays:

$$\theta = 0 \text{ and } \theta = 2\alpha,$$

and similarly, condition (7b) implies that $h = \sigma t$ everywhere along the same lines. With these conditions the problem is solved. To obtain the "free surface" from the solution above, condition (8) is then imposed. That the foregoing does not yield meaningful results can easily be seen from the following. Consider the behaviour of the solution of the boundary value problem given by equations (9), (9a), and (9b) for, say, Case I, i.e.:

$$h\ (r,\ 0,\ t) = h\ (r,\ 2\alpha,\ t) = H,$$

as $t \to \infty$. By physical reasoning or by actually solving the problem, one obtains the result that:

$$\lim_{t\to\infty} h\ (r,\ \theta,\ t) = H \qquad r > 0,\ 0 < \theta < 2\alpha.$$

That is, h goes to H everywhere in the wedge. This equality must hold, in particular, on the "free surface", given by (8), so that the latter reduces to:

$$H = r \sin\ (\alpha - \theta)$$

or

$$H = r\ [\sin \alpha \cos \theta - \cos \alpha \sin \theta] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and since $r = \sqrt{x^2 + y^2}$, $\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$, $\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$ (2) becomes:

$$H = x \sin \alpha - y \cos \alpha$$

so that the "free surface" ultimately becomes a straight line parallel to the sloping surface and a distance H above it (?).

Obviously something is wrong since from physical considerations it is known that the water surface in the embankment ultimately approaches the horizontal. The correct boundary conditions should be step functions as follows:

$$h = \begin{cases} H \text{ for } 0 \leqslant r \leqslant H/\sin \alpha \\ 0 \text{ for } \quad r > H/\sin \alpha \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

and

$$h = \begin{cases} \sigma t \text{ for } 0 \leqslant r \leqslant \sigma t/\sin \alpha \\ 0 \text{ for } \quad r > \sigma t/\sin \alpha, \end{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

indicating that the water level in the reservoir is rising at some predetermined rate and on this basis the water surface in the embankment must be below this surface. Condition (8) of the Paper is not used.

The agreement of the Authors' form of solution with that of Polubarinova-Kochina for the case of $\alpha = 90°$ is quite accidental. Polubarinova-Kochina employs a one-dimensional approach to obtain a solution for the problem of flow out of a ditch. When the Authors' solution is specialised for $\alpha = 90°$ the extraneous co-ordinate drops out altogether also yielding the transient heat equation in one dimension. However, Polubarinova-Kochina's variable is definitely the height to the free surface, and nowhere does she employ an auxiliary condition of the type (8) since, as indicated, this is superfluous. It is noted, moreover, that Polubarinova-Kochina states that the use of this equation in the first place ". . . is permissible (only) if we want to obtain crude results as a guidance," which agrees with the Writer's earlier comments.

Most of the shortcomings discussed, I believe, are caused by the unfortunate representation of h as a "head" rather than as the height to the free surface. With the proper definition of h the analysis presented may be applicable to the "long time" solution of this problem, i.e. when the free surface is nearly horizontal. For this case I believe the one-dimensional approach should be valid. A similar problem involving long-time behaviour has been presented by De Wiest (see reference) for a two-dimensional flow field (flow in a vertical plane). Here the final (steady state) free surface was not horizontal but, in fact, parabolic so that a two-dimensional analysis had to be employed.

Yours faithfully,

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REFERENCE

De Wiest, Roger, 1961. "Free surface flow in homogeneous porous medium." *J. Hyd. Div., Proc. Amer. Soc. civ. Engrs.*, 87:HY4:181–220 (July) 1961.
